Efficient Approximation with Neural Networks: A Comparison of Gate Functions[1]

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Abstract

We compare different gate functions in terms of the approximation power of their circuits. Evaluation criteria are circuit size s, circuit depth d and the approximation error $e(s, d)$. Informally, gate functions γ_1 and γ_2 are called equivalent if $\{\gamma_1\}$-circuits of size s and depth d can be approximated by $\{\gamma_2\}$-circuits (and vice versa) of size poly(s), depth $O(d)$ with approximation error $e(s, d) = 2^{-s}$. Our goal is to determine those gate functions that are equivalent to splines relative to this error model.

The class of equivalent gate functions contains, among others, the exponential function, the natural logarithm, (non-polynomial) rational functions and (non-polynomial) roots. Newman's result, i.e., approximating $| x |$ by rational functions, is obtained as a corollary of this equivalence result. Provably not equivalent are polynomials, the *sine*-function and linear splines.

1 Introduction

We consider efficient approximations of a given multivariate function $f : [-1,1]^n \to \mathcal{R}$ by feedforward neural networks. We first introduce the notion of a feedforward net (or in our terminology, the notion of a Γ-circuit).

Definition 1.1 **(a)** *Let Γ be a class of real-valued functions, where each function is defined on some subset of $\mathcal{R}$. A Γ-circuit C is an unbounded fan-in circuit whose edges and vertices are labeled by rational numbers. We assume that C has a unique sink t. The rational number assigned to an edge (resp. vertex) is called its weight (resp. its threshold). Moreover, to each vertex v, a gate function $\gamma_v \in \Gamma$ is assigned.*
(b) *The circuit C computes a function $F_C : [-1,1]^n \to \mathcal{R}$ as follows. The components of the input vector $x = (x_1, \ldots, x_n) \in [-1,1]^n$ are assigned to the sources of C. Let $v_1, \ldots, v_r$ be the immediate predecessors of a vertex v. The input for v is then $s_v(x) = \sum_{i=1}^{r} w_i y_i - t_v$, where w_i is the weight of the edge (v_i, v), t_v is the threshold of v and y_i is the value assigned to v_i. If v is not the sink (i.e., $v \neq t$), then assign the value $\gamma_v(s_v(x))$ to v; otherwise assign $s_t(x)$ to t. Finally, F_C, the function computed by C, is defined by $F_C = s_t$.*
(c) *The depth of C is the length of the longest path from a source to the sink of C, the size of C is the number of vertices of C which are not sources of C.*

Since the sink computes a weighted sum, it becomes possible to approximate arbitrary functions, although the gate functions in Γ may have a restricted image.

A great deal of work has been done showing that circuits of depth 2 can approximate (in various norms) large function classes (including continuous functions) arbitrarily well (Arai, 1989; Carrol and Dickinson, 1989; Cybenko, 1989; Funahashi, 1989; Gallant and White, 1988; Hornik *et al.* 1989; Irie,1988; Lapades and Farber, 1987; Nielson, 1989; Poggio and Girosi, 1989; Wei *et al.*, 1991). Research on efficient approximations has only started recently (Williamson and Paice, 1991).

Various gate functions have been used, among others, *the cosine squasher, the standard sigmoid, radial basis functions, generalized radial basis functions, polynomials, trigonometric polynomials and binary thresholds.* Still, as we will see, these functions differ significantly in terms of their approximation power.

Our goal is to compare gate functions in terms of *efficiency* and *quality of approximation.* We measure efficiency by the size of the circuit and by its depth. Another resource of interest is *the*

Lipschitz-bound of the circuit, which is a measure of the numerical stability of the circuit.

Definition 1.2 **(a)** *Let* $r = \frac{p}{q}$ *be a rational number with* $(p, q) = 1$. *Then we say that* r *has logarithmic size at most* s, *provided* $|p|, |q| \leq 2^s$.
(b) *A* Γ*-circuit* C *has Lipschitz-bound* L *over a domain* D *if and only if*

- *all weights and thresholds of* C *are rational numbers of logarithmic size at most* $\log_2 L$,
- *for each vertex* v *of* C,
  - $\gamma_v(s_v(D)) \subseteq [-L, L]$ *and*
  - *the gate function* γ_v *has Lipschitz-bound at most* L *for all inputs from*

$$\bigcup_{y \in s_v(D)} [y-1, y+1].$$

(Thus we do not demand that gate function γ_v has Lipschitz-bound L, but only that γ_v has Lipschitz-bound L for the inputs it receives. Moreover, the actually received inputs have to be bounded away from regions with higher Lipschitz-bounds.) We measure the quality of an approximation of function f by function g by the Chebychev norm:

Definition 1.3 *Let* $f, g : D \to \mathcal{R}$ *be real-valued functions over the domain* $D \subseteq \mathcal{R}^n$. *We set*

$$||f - g||_D = \sup\{|f(x) - g(x)| \ : \ x \in D\}.$$

Let Γ be a class of gate functions. We are mainly interested in the following question.

> Given two classes Γ_1 and Γ_2 of gate functions, when do Γ_1-circuits and Γ_2-circuits have *essentially* the same "approximation power" with respect to error 2^{-s}?

We formalize the notion of *having essentially the same approximation power* as follows.

Definition 1.4 *Let* Γ_1 *and* Γ_2 *be classes of gate functions.*
(a) *We say that* Γ_2 *simulates* Γ_1 *(denoted by* $\Gamma_1 \leq \Gamma_2$*) if and only if there is a constant* $k \geq 1$ *such that*

> *for all* Γ_1*-circuits* C_1 *of size at most* s, *depth at most* d *and Lipschitz-bound* 2^s *over* $[-1, 1]^n$ *there is a* Γ_2*-circuit* C_2 *of size at most* $(s+1)^k$, *depth at most* $k \cdot (d+1)$ *and Lipschitz-bound* $2^{(s+1)^k}$ *over* $[-1, 1]^n$ *with*
>
> $$||F_{C_1} - F_{C_2}||_{[-1,1]^n} \leq 2^{-s}.$$

(b) *We say that Γ_1 and Γ_2 are equivalent (denoted by $\Gamma_1 \equiv \Gamma_2$) if and only if $\Gamma_1 \leq \Gamma_2$ and $\Gamma_2 \leq \Gamma_1$.*

In other words, when simulating classes of gate functions, we allow depth to increase by a constant factor, size and the logarithm of the Lipschitz-bound to increase polynomially. The relatively large Lipschitz-bounds should not come as a surprise, since the negative exponential error 2^{-s} requires correspondingly large weights for the simulating circuit.

Proposition 1.1 *$\equiv$ is an equivalence relation on classes of gate functions.*

Proof. We observe first that $\equiv$ is symmetric and reflexive. It remains to verify that $\equiv$ is also transitive. So assume that $\Gamma_1 \equiv \Gamma_2$ and $\Gamma_2 \equiv \Gamma_3$.

In particular, let C_1 be a Γ_1-circuit of size at most s, depth at most d and Lipschitz-bound at most 2^s over $[-1,1]^n$. Since $\Gamma_1 \leq \Gamma_2$, there is a constant $k_1 \geq 1$ (independent of C_1) and a Γ_2-circuit C_2 (of size at most $s' = (s+2)^{k_1}$, depth at most $d' = k_1 \cdot (d+1)$ and Lipschitz-bound at most $2^{s'}$ over $[-1,1]^n$) such that

$$||F_{C_1} - F_{C_2}|| \leq 2^{-(s+1)} = \frac{1}{2} \cdot 2^{-s}.$$

Moreover, since $\Gamma_2 \leq \Gamma_3$, there is a constant $k_2 \geq 1$ (independent of C_2) and a Γ_3-circuit C_3 (of size at most $s'' = (s'+1)^{k_2}$, depth at most $d'' = k_2 \cdot (d'+1)$ and Lipschitz-bound at most $2^{s''}$ over $[-1,1]^n$) such that

$$||F_{C_2} - F_{C_3}|| \leq 2^{-s'} \leq 2^{-(s+2)^{k_1}} \leq \frac{1}{2} \cdot 2^{-s}.$$

Thus

$$||F_{C_1} - F_{C_3}|| \leq 2^{-s},$$

where C_3 has size

$$s'' = (s'+1)^{k_2} = ((s+2)^{k_1}+1)^{k_2} \leq ((3 \cdot s)^{k_1} + (3 \cdot s)^{k_1})^{k_2} = (2 \cdot (3 \cdot s)^{k_1})^{k_2} \leq (s+1)^{\kappa + k_1 \cdot k_2},$$

if κ is chosen sufficiently large. Moreover, the depth of C_3 is bounded by

$$k_2 \cdot (d'+1) = k_2 \cdot (k_1 \cdot (d+1) + 1) \leq 2k_2 \cdot k_1 \cdot (d+1).$$

Thus, for $k = \max\{\kappa + k_1 \cdot k_2, 2k_2 \cdot k_1\}$, C_3 has size at most $(s+1)^k$, depth at most $k \cdot (d+1)$, Lipschitz-bound at most $2^{s''} \leq 2^{(s+1)^k}$ over $[-1,1]^n$. Moreover C_3 approximates C_1 with error at most 2^{-s}. Thus we have obtained that $\Gamma_1 \leq \Gamma_3$. The claim follows, since $\Gamma_3 \leq_e \Gamma_1$ can be shown analogously. □

We are particularly interested in the approximation power of spline circuits.

Definition 1.5 **(a)** *A continuous function $f : \mathcal{R} \to \mathcal{R}$ is called a k-spline of degree δ if and only if there are knots $-\infty = x_0 < x_1 < \ldots < x_k < x_{k+1} = \infty$ and polynomials $p_1, \ldots, p_{k+1}$ of degree at most δ such that $f(x) = p_r(x)$ for all x with $x \in [x_{r-1}, x_r]$.*

(b) *A spline circuit (resp. polynomial circuit) C of size s and weight w over a domain $D \subseteq \mathcal{R}$ is an unbounded fan-in circuit with s vertices. The gate functions of C are 1-splines (resp. univariate polynomials)) of degree at most s. Moreover C has Lipschitz-bound at most 2^w over D and the coefficients of all involved polynomials, as well as the weights and thresholds of C are rational numbers of logarithmic size at most w.*

(c) *We use the notation*

$$\Gamma \leq \textit{splines}$$

to express that there is a constant c such that any Γ-circuit of size at most s, depth at most d and Lipschitz-bound at most 2^s over $[-1,1]^n$ can be approximated by a spline circuit of size at most $(s+1)^c$, depth at most $c \cdot (d+1)$ and weight at most $(s+1)^c$. The notation

$$\textit{splines} \leq \Gamma \quad \textit{and} \quad \textit{splines} \equiv \Gamma$$

is introduced analogously.

We can now introduce our main result.

Theorem 1.1 *The following gate functions are equivalent to splines.*

- **(a)** *the exponential function,*
- **(b)** *the natural logarithm,*
- **(c)** *any rational function which is not a polynomial,*
- **(d)** *any root x^α, provided α is not a natural number,*
- **(e)** *the standard sigmoid $\sigma(x) = \frac{1}{1+e^{-x}}$.*

This result is the consequence of identifying a large class of gate functions (i.e., powerful gate functions, see Definition 3.1), which have *at least* the approximation power of splines (see Theorem 3.1). Any powerful gate function γ has to satisfy two properties. Firstly, γ has to have a convergent Taylor-series in some small interval; in this case, polynomials can be approximated tightly (Proposition 2.4). Secondly, $\{\gamma\}$-circuits have to be capable of approximating the binary threshold $t(x) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$ (over the interval $[-1,1] - [-2^{-s}, 2^{-s}]$) with size polynomial in s, constant depth and error 2^{-s}.

In Definition 3.1, we describe a Lipschitz-type property that guarantees a tight approximation of the binary threshold. This property can be checked quite easily (Lemma 3.1) and Theorem 1.1 becomes a corollary of Theorem 3.1. As a second consequence of Theorem 3.1, we obtain Newman's approximation result of $|\, x \,|$ by rational functions (Newman, 1964) (see Remark 3.1).

Notable exceptions from the list of functions equivalent to splines are polynomials, trigonometric polynomials and linear splines. Linear splines and polynomials are properly weaker than splines (Proposition 3.1). In Proposition 3.2 we show that spline circuits of depth d require size $s = A^{\Omega(1/d)}$ to approximate $sine(Ax)$ with error of at most 1/2. Thus *sine* and splines are not equivalent. On the other hand, spline circuits of constant depth and size polynomial in $\log A$ are capable of approximating $sine(Ax)$ tightly, *provided* the input x is given *in binary* (Reif, 1987). Therefore, the complexity of extracting bits from the given analog input is high as well.

A related question is the computing power of gate functions for binary input. In (DasGupta and Schnitger, 1995) we show that the language family

$$L_n = \{(x_1, \ldots, x_n, y_1, \ldots, y_{n^2}) \;:\; (\sum_{i=1}^{n} x_i)^2 \geq \sum_{i=1}^{n^2} y_i \;\}$$

can be recognized by $\{\gamma\}$-circuits with two gates, whereas threshold circuits require size $\Omega(\log_2 n)$. (γ has to be three times continuously differentiable in some small neighborhood of 0 and $\gamma^{(2)}(0) \neq 0$.) Hence real-valued gate functions can be considerably more powerful.

We begin the next section by showing that it suffices to tightly approximate single gate functions in Γ_1 by small Γ_2-circuits of constant depth in order to establish $\Gamma_1 \leq \Gamma_2$. In section 2.1 we review the approximation power of spline-circuits and verify that each function mentioned in Theorem 1.1 can be efficiently simulated by spline circuits. Section 2.2 shows how to approximately compute polynomials assuming a sufficiently smooth gate function. Approximations of the binary threshold are described in the section 3. Finally, a summary is given in section 4. A preliminary version of this paper appears in (DasGupta and Schnitger, 1993).

2 Simulating gate functions

We first observe that an approximation of individual gate functions establishes an approximation of the entire circuit.

Definition 2.1 **(a)** *For a subset $D \subseteq \mathcal{R}$ we set*

$$\overline{D} = \bigcup_{y \in D} [y-1, y+1].$$

(b) *Let γ be a gate function and let Γ be a class of gate functions. For a natural number k we say that Γ k-simulates γ, if and only if*

> *for each integer $s \geq k$ and any subset $D(\gamma) \subseteq [-2^{3s}, 2^{3s}]$ such that γ has Lipschitz-bound at most 2^s over $\overline{D(\gamma)}$*

there is

> *a Γ-circuit $C(\gamma, s)$ of size at most $(s+1)^k$ and depth at most k which approximates γ over $\overline{D(\gamma)}$ with approximation error at most 2^{-s}. Moreover we demand that $C(\gamma, s)$ has Lipschitz-bound at most $2^{(s+1)^k}$ over $\overline{D(\gamma)}$.*

Proposition 2.1 *Let Γ_1 and Γ_2 be classes of gate functions and let k be a natural number. Assume that each function $\gamma \in \Gamma_1$ can be k-simulated by Γ_2. Then*

$$\Gamma_1 \leq \Gamma_2.$$

Proof. Let C_1 be an arbitrary Γ_1-circuit with n inputs and assume that C_1 has depth at most d, size at most s and Lipschitz-bound at most 2^s over $[-1, 1]^n$.

Let $s_v(x_1, \ldots, x_n)$ be the input received by a vertex v of C_1 for circuit input $(x_1, \ldots, x_n)$ (see Definition 1.1). Assume that $s_v(x_1, \ldots, x_n) = \sum_{i=1}^m w_{v,i} \cdot y_i - t_v$, with gate inputs $y_1, \ldots, y_m$. We set $D(\gamma_v) = s_v([-1, 1]^n)$ for the gate function γ_v of vertex v. Since C_1 has Lipschitz-bound 2^s over $[-1, 1]^n$, the gate function γ_v of v has Lipschitz-bound 2^s over the domain $\overline{D(\gamma_v)}$. Moreover each gate of C_1 outputs a result of absolute value at most 2^s. Hence $|s_v(x_1, \ldots x_n)| \leq \sum_{i=1}^m |w_{v,i}| \cdot |y_i| + |t_v| \leq s \cdot 2^s \cdot 2^s + 2^s \leq 2^{3s}$ and $D(\gamma_v) = s_v([-1, 1]^n)$ is contained in the interval $[-2^{3s}, 2^{3s}]$.

By assumption, γ_v can be k-simulated by a Γ_2-circuit $C(\gamma_v, 5 \cdot s^2)$. $C(\gamma_v, 5 \cdot s^2)$ has depth at most k, size at most $(5 \cdot s^2 + 1)^k \leq (s+1)^{4k}$ and Lipschitz-bound at most $2^{(s+1)^{4k}}$ over $\overline{D(\gamma_v)}$. We replace gate γ_v by γ_v^*, the function computed by $C(\gamma_v, 5 \cdot s^2)$ and keep the weights $w_{v,i}$ and the threshold t_v. We obtain a Γ_2-circuit C_2 of depth at most $k \cdot d$ and size at most $s \cdot (s+1)^{4k} \leq (s+1)^{4k+1}$. We have to verify that C_2 meets the approximation and Lipschitz requirements.

If the gate inputs $y_1, \ldots, y_m$ are approximated by $y_1^*, \ldots, y_m^*$ with respective error at most ε, then we obtain

$$|(\sum_{i=1}^m w_{v,i} \cdot y_i - t) - (\sum_{i=1}^m w_{v,i} \cdot y_i^* - t)| \leq \sum_{i=1}^m |w_{v,i}| \cdot \varepsilon \leq s \cdot 2^s \cdot \varepsilon \leq 2^{2s} \cdot \varepsilon.$$

In particular, $\sum_{i=1}^m w_{v,i} \cdot y_i^* - t \in \overline{D(\gamma_v)}$, provided $\varepsilon \le 2^{-2s}$. Remember that $\gamma_v(x)$ is approximated by $\gamma_v^*(x)$ with error at most 2^{-5s^2} for $x \in \overline{D(\gamma_v)}$ and hence

$$|\gamma_v(\sum_{i=1}^m w_{v,i} \cdot y_i^* - t) - \gamma_v^*(\sum_{i=1}^m w_{v,i} \cdot y_i^* - t)| \le 2^{-5s^2}.$$

Thus, assuming $\varepsilon \le 2^{-2s}$ and utilizing that γ_v has Lipschitz-bound 2^s over $\overline{D(\gamma_v)}$, $\gamma_v(\sum_{i=1}^m w_{v,i} \cdot y_i - t_v)$ is approximated with error at most

$$\begin{aligned}
& |\gamma_v(\sum_{i=1}^m w_{v,i} \cdot y_i - t) - \gamma_v^*(\sum_{i=1}^m w_{v,i} \cdot y_i^* - t)| \\
\le\ & |\gamma_v(\sum_{i=1}^m w_{v,i} \cdot y_i - t) - \gamma_v(\sum_{i=1}^m w_{v,i} \cdot y_i^* - t)| + |\gamma_v(\sum_{i=1}^m w_{v,i} \cdot y_i^* - t) - \gamma_v^*(\sum_{i=1}^m w_{v,i} \cdot y_i^* - t)| \\
\le\ & 2^s \cdot |(\sum_{i=1}^m w_{v,i} \cdot y_i - t) - (\sum_{i=1}^m w_{v,i} \cdot y_i^* - t)| + 2^{-5s^2} \\
\le\ & 2^s \cdot 2^{2s} \cdot \varepsilon + 2^{-5s^2}.
\end{aligned}$$

Hence, if ε_i is an upper bound for the approximation error of gates of depth i, we obtain the recurrence

$$\varepsilon_0 = 0 \text{ and } \varepsilon_{i+1} = 2^{3s} \cdot \varepsilon_i + 2^{-5s^2}.$$

Thus the overall approximation error of C_2 is bounded by

$$\varepsilon_d \le \varepsilon_s = 2^{-5s^2} \cdot (\sum_{i=0}^{s-1} 2^{3si}) \le 2^{-5s^2} \cdot 2^{3s^2} \le 2^{-2s^2} \le 2^{-2s}$$

and $\varepsilon_i \le 2^{-2s}$ holds throughout.

The claim follows, once we have shown that C_2 has Lipschitz-bound $2^{(s+1)^{4k}}$ over $[-1,1]^n$. By construction, all weights and thresholds of C_2 are upper-bounded by $2^{(s+1)^{4k}}$. Each inserted subcircuit $C(\gamma_v, 5 \cdot s^2)$ receives only inputs from $\overline{D(\gamma_v)}$ and it has Lipschitz-bound $2^{(s+1)^{4k}}$ over $\overline{D(\gamma_v)}$. Hence the Lipschitz-bound $2^{(s+1)^{4k}}$ follows for C_2 as well. □

2.1 Approximating with Splines

In Definition 1.5 we have introduced splines and spline-circuits. Remember that a spline circuit of size s is built from s 1-splines of degree at most s; the unique sink computes the weighted sum of its inputs.

Remark 2.1 *Let C be a spline circuit of size s, depth d and weight at most w over a subset $D \subseteq \mathcal{R}$. An inductive argument shows that C computes a s^{d-1}-spline F_C of degree at most s^{d-1}. Moreover, all coefficients have logarithmic size at most $w' = O(w \cdot s^d)$.*

Now assume that F_C is represented by the polynomials $p_1, \ldots, p_{c+1}$ and the knots $x_1, \ldots, x_c$, where $c = s^{d-1}$. We introduce the 1-splines $f_1, \ldots, f_c$ and the polynomial $f_{c+1} = p_{c+1}$, where $f_i(x) = p_i(x) - p_{i+1}(x)$ for $x \leq x_i$ and $f_i(x) = 0$ for $x > x_i$. Then obviously $F_C = \sum_{i=1}^{c+1} f_i$.

The absolute value of a denominator (appearing in f_i) can grow by at most a square (compared to p_i), the absolute value of a numerator can additionally double. Thus $2 \cdot w' + 1 = O(w \cdot s^d)$ is an upper bound on the weight of the circuit.

Hence F_C, the function computed by C, can be computed by a spline circuit of depth 2, size $s^{d-1} + 2$ and weight at most $O(w \cdot s^d)$ over domain D.

Splines are quite powerful. In particular they achieve tight approximations whenever the domain can be partitioned into not "too many" intervals where the function in question has a sufficiently fast converging Taylor series:

Proposition 2.2 *For each of the following functions and domains D an integer S exists such that for any $s \geq S$*

> *spline circuits of depth 2, size and weight polynomial in s approximate with error at most 2^{-s} over the domain D,*

(a) *the exponential function over the domain $D = [-\infty, s]$,*
(b) *the natural logarithm over the domain $D = [2^{-s}, 2^s]$,*
(c) *any rational function r over the domain $D = [-2^s, 2^s]$, except for intervals of length 1 around the poles of r,*
(d) *any root x^α over the domain $D = [0, 2^s]$,*
(e) *the standard sigmoid σ over the domain $D = [-\infty, \infty]$.*

Proof. We begin with two initial remarks. Firstly, a spline circuit of depth 2 and size polynomial in s has weight polynomial in s, provided the coefficients of all involved polynomials and the weights and thresholds of the circuit are rational numbers of logarithmic size at most w.

Secondly assume that, for an interval I with $a \in I$, the function $f : I \to \mathcal{R}$ has the convergent Taylor series $f(x) = \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} f^{(k)}(a)$. Then $T_s(x) = \sum_{k=0}^{s} \frac{(x-a)^k}{k!} f^{(k)}(a)$ is the Taylor polynomial of f of degree s. We obtain, as a consequence of the Lagrange error estimate,

$$|f(x) - T_s(x)| = \frac{|x-a|^{s+1}}{(s+1)!} f^{(s+1)}(\xi_x) \quad \text{for some } \xi_x \in I.$$

(a) We approximate $\exp(x)$ over the domain $[-s, s]$ by its Taylor polynomial $T_{s^2}(x) = \sum_{k=0}^{s^2} \frac{x^k}{k!}$. The resulting approximation error will be at most $s^{s^2+1} \cdot \exp(s)/(s^2+1)! \leq 2^{-2s}$, provided s is sufficiently large.

Over the domain $[-\infty, -s]$, we approximate by the constant $y = T_{s^2}(-s)$. This time the approximation error is bounded by $\exp(-s) + 2^{-2s} \leq 2^{-s}$ (for $s \geq 2$). Hence, we can approximate $\exp(s)$ by a 1-spline of degree at most s^2.

(b) Let a be a positive real number and set $D(a) = [\frac{a}{2}, \frac{3 \cdot a}{2}]$. For $x \in D(a)$ we obtain the representation

$$\ln(x) = \ln(a) + \sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1} (\frac{x-a}{a})^{k+1}.$$

The Taylor polynomial $T_{r,a}(x) = \ln(a) + \sum_{k=0}^{r-1} (-1)^k \frac{1}{k+1} (\frac{x-a}{a})^{k+1}$ approximates $\ln(x)$ with error at most 2^{-r} over the domain $D(a)$, since

$$|\ln(x) - T_{r,a}(x)| = |\sum_{k=r}^{\infty} (-1)^k \frac{1}{k+1} (\frac{x-a}{a})^{k+1}| \leq \sum_{k=r}^{\infty} \frac{1}{k+1} (\frac{|x-a|}{|a|})^{k+1} \leq \sum_{k=r+1}^{\infty} \frac{1}{2^k} = 2^{-r}.$$

To obtain the required spline circuit, divide the interval $[2^{-s}, 2^s]$ into the intervals

$$[2^{-s}, 2^{-(s-1)}], [2^{-(s-1)}, 2^{-(s-2)}], \ldots, [2^{s-2}, 2^{s-1}], [2^{s-1}, 2^s]$$

and approximate $\ln(x)$ by the Taylor polynomial $T_{3s,2^i}(x)$ over the interval $[2^{i-1}, 2^i] \subseteq [\frac{1}{2} \cdot 2^i, \frac{3}{2} \cdot 2^i]$. The final approximation error will be bounded by 2^{-3s}.

But the resulting spline is not continuous, since in general $T_{3s,2^i}(2^i) \neq T_{3s,2^{i+1}}(2^i)$. This is easily corrected by creating the new knots $2^i - 2^{-s^2}$, $2^i + 2^{-s^2}$ and approximating with $T_{3s,2^i}(2^i)$ (over the interval $[2^{i-1} + 2^{-3s}, 2^i - 2^{-3s}]$) followed by a linear spline (over the interval $[2^i - 2^{-3s}, 2^i + 2^{-3s}]$).

Observe that $\ln(x+y) - \ln(x) \leq \frac{y}{x}$ for $x, y > 0$. Since the logarithm is an increasing function and since $T_{3s,2^i}$ and $T_{3s,2^{i+1}}$ approximate with error at most 2^{-3s}, the linear spline will approximate with error at most

$$\ln(2^i + 2^{-3s}) - \ln(2^i - 2^{-3s}) + 2^{-3s} \leq \frac{2 \cdot 2^{-3s}}{2^i - 2^{-3s}} + 2^{-3s} \leq \frac{2 \cdot 2^{-3s}}{2^{-s} - 2^{-3s}} + 2^{-3s} \leq 2^{-s},$$

for sufficiently large s. Thus we have approximated $\ln(x)$ by an $O(s)$-spline of degree $O(s)$. Hence, with Remark 2.1, we can therefore approximate $\ln(x)$ by a spline circuit of depth 2, size and weight polynomial in s and error at most 2^{-s}. (Observe that our splines have real-valued coefficients, weights and thresholds of small absolute value. But since we only consider the domain $[2^{-s}, 2^s]$, a transition to rational coefficients, weights and thresholds of small logarithmic size is immediate.)

(c) Let $r(x) = \frac{p(x)}{q(x)}$ be a rational function. Choose S sufficiently large such that for all $s \geq S$,

- the degree of p and the degree of q is at most S,
- all coefficients of r have size at most 2^S and
- $2^{-s^2} \leq |q(x)| \leq 2^{s^2}$, whenever x has distance at least 1 from a zero of q.
- $|p(x)| \leq 2^{s^2}$ for $x \in [-2^s, 2^s]$

Observe that $\frac{1}{q(x)} = sign(q(x)) \cdot \exp(-\ln(|q(x)|)\ )$. Let $\exp^*$ be the spline-approximation of exp (with approximation error at most 2^{-s^3} over the domain $[-s^2, s^2]$) obtained in part (a) and let $\ln^*$ be the spline-approximation of ln (with approximation error at most 2^{-s^3} over the domain $[2^{-s^2}, 2^{s^2}]$) obtained in part (b). Observe that both spline approximations require only size polynomial in s and depth 2. We approximate $r(x)$ as follows.

(i) Compute $|q(x)|$ *exactly* by computing either $q(x)$ or $-q(x)$ (as required) between successive real zeroes of $q(x)$. Thus $|q|$ can be computed by a S-spline of degree S.

(ii) Approximate $\frac{1}{q(x)}$ by $sign(q(x)) \cdot \exp^*(\ -\ln^*(|q(x)|)\ )$ between successive real zeroes of $q(x)$ and connect successive polynomials by linear splines.

(iii) Compute the final approximation $p(x) \cdot sign(q(x)) \cdot \exp^*(\ -\ln^*(|q(x)|)\ )$.

Observing $2^{-s^2} \leq |q(x)| \leq 2^{s^2}$, we obtain for the approximation error in steps (i) and (ii),

$$\begin{aligned} |\exp(-\ln(|q(x)|)\ ) - \exp^*(-\ln^*(|q(x)|)\ )| &\leq |\exp(-\ln(|q(x)|)\ ) - \exp(-\ln^*(|q(x)|)\ )| + 2^{-s^3} \\ &\leq |\exp(-\ln(|q(x)|)\ ) - \exp(-\ln(|q(x)|)\ +\varepsilon)| + 2^{-s^3} \end{aligned}$$

with $|\varepsilon| \leq 2^{-s^3}$. Since $|1 - \exp(x)| \leq |x| \cdot \exp(1)$ for $x \in [0, 1]$, we obtain,

$$\begin{aligned} |\exp(-\ln(|q(x)|)\ ) - \exp^*(-\ln^*(|q(x)|)\ )| &\leq \exp(-\ln(|q(x)|)\ ) \cdot |1 - \exp(\varepsilon))| + 2^{-s^3} \\ &\leq 2^{s^2} \cdot \varepsilon \cdot \exp(1) + 2^{-s^3} \leq 2 \cdot \exp(1) \cdot 2^{s^2 - s^3}. \end{aligned}$$

Hence the overall approximation error is bounded by $2 \cdot \exp(1) \cdot 2^{2s^2 - s^3}$, since $|p(x)| \leq 2^{s^2}$. By Remark 2.1 we obtain a spline circuit of depth 2, polynomial size and weight, since the circuit constructed so far has bounded depth.

(d) To approximate x^α, we approximate $\exp(\alpha \ln(x))$ for $x \in [\exp(-s/\alpha), 2^s]$ using a construction similar to part (c). Observe that $0 \leq x^\alpha < \exp(-s)$ for $x < \exp(-s/\alpha)$ and it suffices to continue the approximation by a constant over $[0, \exp(-s/\alpha]$.
(e) For $x \in [-2s, 2s]$, approximate the standard sigmoid $\sigma(x) = \frac{1}{1+\exp(-x)}$ using part (a) and (c). Obviously, an overall approximation error of at most $2^{-(s+1)}$ can be reached by a spline circuit of depth 2 and size polynomial in s. Let a_{-2s} (resp. a_{2s}) be the value of the approximating function $x = -2s$ (resp. $x = 2s$).

Approximate $\sigma(x)$ over the domain $]-\infty, -2s]$ (resp. $[2s, \infty[$) by $y = a_{-2s}$ (resp. $y = a_{2s}$). Observe that for $x \leq -2s$,

$$\begin{aligned} |\sigma(x) - a_{-2s}| &\leq |\sigma(x) - \sigma(-2s)| + |\sigma(-2s) - a_{-2s}| \\ &\leq \sigma(-2s) + 2^{-(s+1)} < \exp(-2s) + 2^{-(s+1)} < 2^{-s} \end{aligned}$$

and for $x \geq 2s$,

$$\begin{aligned} |\sigma(x) - a_{2s}| &\leq |\sigma(x) - \sigma(2s)| + |\sigma(2s) - a_{2s}| \\ &\leq |1 - \sigma(2s)| + 2^{-(s+1)} = \frac{\exp(-2s)}{1 + \exp(-2s)} + 2^{-(s+1)} \leq \exp(-2s) + 2^{-(s+1)} < 2^{-s}. \end{aligned}$$

□

We can now combine Propositions 2.1 and 2.2 to formally verify the approximation power of splines in our setting.

Lemma 2.1 *For each of the gate functions γ mentioned below we have $\{\gamma\} \leq$ splines.*
(a) $\gamma(x) = \exp(x)$,
(b) $\gamma(x) = \ln(x)$,
(c) $\gamma(x) = r(x)$, *where r is a rational function,*
(d) $\gamma(x) = x^\alpha$,
(e) $\gamma(x) = \frac{1}{1+\exp(-x)}$,

Proof. Proposition 2.2 constructs approximations with error at most 2^{-s} for all involved functions. However the domain of approximation differs. To apply Proposition 2.1 we have to show in each case that the domain D of approximation is sufficiently large. In particular Proposition 2.1 applies the restriction that the gate function γ has Lipschitz-bound at most 2^s over $\overline{D} = \bigcup_{y \in D}[y-1, y+1]$ and hence it suffices to show that $|\gamma'(x)| > 2^s$ for $x \notin [-2^{3s}, 2^{3s}] - \overline{D}$.

Proposition 2.2 provides sufficiently large domains of for all functions with the exceptions of $\gamma(x) = x^\alpha$, since we only approximate over $D = [0, 2^s]$. But, if $(-1)^\alpha$ is a real number, then we can easily extend the domain of approximation to $[-2^{3s}, 2^{3s}]$. □

2.2 Approximating Polynomials

We show next that circuits composed of sufficiently smooth gate functions are capable of approximating polynomials within any degree of accuracy.

Definition 2.2 *Let $\gamma : \mathcal{R} \to \mathcal{R}$ be a function. We call γ non-trivially smooth with parameter k if and only if there exists rational numbers α, β ($\alpha > 0$) and an integer k such that α and β have logarithmic size at most k and*
(a) *γ can be represented by the power series $\sum_{i=0}^{\infty} a_i(x-\beta)^i$ for all $x \in [\beta - \alpha, \beta + \alpha]$. For each $i > 1$, a_i is a rational number of logarithmic size at most i^k.*
(b) *For each $i > 1$ there exists j with $i \leq j \leq i^k$ and $a_j \neq 0$.*
(c) *For each $i > 1$, $||\gamma^{(i)}||_{[-\alpha,\alpha]} \leq 2^{i^k}$.*

The following result follows from the fact that the monomials x^{k_1}, x^{k_2},…,x^{k_n} form a Chebycheff system over $[0, \infty]$ for n pairwise distinct positive integers $k_1, k_2, \ldots k_n$(see page 9 in (Karlin and Studden, 1966)).

Fact 2.1 *The $n \times n$ matrix $X = (\ x_i^{k_j}\ )$ is non-singular for any collection of pairwise distinct positive reals $x_1, x_2, \ldots, x_n$ and n pairwise distinct positive integers $k_1, k_2, \ldots, k_n$.*

We also need the following result.

Proposition 2.3 **(a)** *Let $Ax = b$ be a linear system with a non-singular $n \times n$ matrix A. Assume that all entries of A (resp. all components of b) are rational numbers with logarithmic size at most $l_1 \geq 1$ (resp. $l_2 \geq 1$).*

Then the system has a solution vector $x = (x_1, x_2, \ldots, x_n)$, where each x_i is a rational number of logarithmic size at most $l_2 + l_1 \cdot poly(n)$.
(b) *The polynomials $(x+1)^n, \ldots, (x+n)^n, (x+n+1)^n$ are linearly independent.*
(c) *Let $p(x)$ be a degree n polynomial whose coefficients are rational numbers of logarithmic size at most max. Then there are rational numbers $\alpha_1, \ldots, \alpha_{n+1}$, of logarithmic size at most max+poly(n), such that*

$$p(x) = \sum_{i=1}^{n+1} \alpha_i (x+i)^n.$$

Proof. **(a)** is an immediate consequence of Cramer's rule.

(b) Consider the equality $\sum_{j=0}^{n} \alpha_j (x+1+j)^n = 0$. Since $(x+1+j)^n = \sum_{i=0}^{n} \binom{n}{i}(1+j)^{n-i} \cdot x^i$, we obtain the equations $\sum_{j=0}^{n} \alpha_j \binom{n}{i}(1+j)^{n-i} = 0$ after equating coefficients. Therefore the above equality leads to the linear system $A\alpha = 0$, where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)^T$ and A is the $(n+1)\times(n+1)$ matrix with $A[i,j] = \binom{n}{i}(1+j)^{n-i}$ (for $0 \leq i, j \leq n$). We have to show that A is non-singular.

We divide row i by $\binom{n}{i}$ and interchange rows i and $n-i$. The resulting matrix B satisfies $B[i,j] = (1+j)^i$ and its non-singularity follows with Fact 2.1.

(c) Using part (b), we find a vector $\alpha = (\alpha_1, \cdots, \alpha_{n+1})$ with

$$p(x) = \sum_{i=1}^{n+1} \alpha_i (x+i)^n.$$

Moreover, α is determined by the linear system

$$A \cdot \alpha = \beta,$$

where the matrix A was introduced in part (b) and β is the vector of coefficients of the polynomial p. The claim follows now with part (a). □

Proposition 2.4 *Assume that γ is non-trivially smooth with parameter k. Let $p(x)$ be a degree n polynomial whose coefficients are rational numbers of logarithmic size at most max.*

Then $p(x)$ can be ε-approximated (over the domain $[-D, D]$ with $[\beta-\alpha, \beta+\alpha] \subseteq [-D, D]$) by a $\{\gamma\}$-circuit C_p. C_p has depth 2 and size $O(n^{2k})$. The Lipschitz-bound of C_p (over $[-D, D]^n$) is at most

$$c_\gamma \cdot [2^{max} \cdot (2+D) \cdot \frac{1}{\varepsilon})]^{poly(n)}.$$

The constant c_γ depends only on γ and not on p.

Proof. We assume, without loss of generality, that $\beta = 0$ and hence that $\gamma(x) = \sum_{i=0}^{\infty} a_i x^i$ for all $x \in [-\alpha, \alpha]$. Let p be a polynomial of degree n. By assumption on γ, there exists an N such that $a_N \neq 0$ and $n \leq N \leq n^k$. We show first that x^N can be approximately computed over the interval $[-\alpha, \alpha]$ and extend this approximation to an approximation over the interval $[-D, D]$. Once this is done, we apply Proposition 2.3 (c) to approximate $p(x)$.

Let $T_N(x) = \sum_{i=0}^{N} a_i x^i$ denote the N'th degree Taylor polynomial of γ. Assume that T_N has s non-zero coefficients, say $a_{k_1}, \ldots, a_{k_s}$ (with $k_s = N$).

Consider the matrix $A = (a_{k_j}(1/i)^{k_j})$. After dividing column j by a_{k_j}, we apply Fact 2.1 and obtain that A is non-singular (and hence the polynomials $T_N(x/i)$ $(1 \leq i \leq s)$ are linearly independent). We apply Proposition 2.3 (a) to the system $A \cdot c = (0, \ldots, 0, 1)$ and obtain that the components of c are rational numbers of logarithmic size at most $poly(N)$. Obviously

$$\sum_{i=1}^{s} c_i T_N(x/i) = x^N.$$

But we have to compute x^N from γ (instead of T_N). Let $L > 1$ be an integer and compute

$$\gamma_N(x) = L^N \sum_{i=1}^{s} c_i \gamma(\frac{x}{i \cdot L}). \tag{1}$$

Observe that for $-\alpha \le x \le \alpha$,

$$\begin{aligned} \gamma_N(x) &= L^N \cdot \sum_{i=1}^{s} c_i T_N(\frac{x}{i \cdot L}) + L^N \sum_{i=1}^{s} c_i \cdot (\gamma(\frac{x}{i \cdot L}) - T_N(\frac{x}{i \cdot L})) \\ &= x^N + L^N \sum_{i=1}^{s} c_i \cdot (\gamma(\frac{x}{i \cdot L}) - T_N(\frac{x}{i \cdot L})). \end{aligned}$$

We claim that γ_N approximates x^N over the interval $[-\alpha, \alpha]$, where the approximation quality will increase with increasing L. Thus we have to determine the approximation error

$$e = \sup\{|L^N \sum_{i=1}^{s} c_i \cdot (\gamma(\frac{x}{i \cdot L}) - T_N(\frac{x}{i \cdot L}))| \ : \ x \in [-\alpha, \alpha]\}.$$

Set $c_{max} = \max\{|c_i| \ : \ 1 \le i \le s\}$. To bound the error, we apply the Lagrange error estimate to each term $\gamma(\frac{x}{i \cdot L}) - T_N(\frac{x}{i \cdot L})$ and obtain that

$$e \le N \cdot L^N \cdot c_{max} \cdot \frac{||\gamma^{(N+1)}||_{[-\alpha,\alpha]}}{(N+1)!} \cdot (\frac{|x|}{L})^{N+1}.$$

Since $D \ge \alpha$,

$$e \le c_{max} ||\gamma^{(N+1)}||_{[-\alpha,\alpha]} \frac{|x|^{N+1}}{L} \le c_{max} ||\gamma^{(N+1)}||_{[-\alpha,\alpha]} \frac{D^{N+1}}{L} \le \frac{(2+D)^{poly(N)}}{L} ||\gamma^{(N+1)}||_{[-\alpha,\alpha]}.$$

Let δ be a positive real number. Then, for

$$L = (2+D)^{poly(n)} \cdot ||\gamma^{(N+1)}||_{[-\alpha,\alpha]} \cdot \frac{1}{\delta \cdot \varepsilon}$$

and $x \in [-\alpha, \alpha]$, we can approximate x^N with error $\delta \cdot \varepsilon$. Hence we can also approximate x^N over the interval $[-(D+N+1), D+N+1]$ by computing

$$\frac{D+N+1}{\alpha})^N \gamma_N(\frac{\alpha}{D+N+1} x).$$

The approximation error will be at most $\frac{D+N+1}{\alpha})^N \cdot \delta \cdot \varepsilon$.

Thus, we obtain also approximations of $(x+1)^N, \ldots, (x+N)^N, (x+N+1)^N$ over the interval $[-D, D]$ with the same approximation error. But, according to Proposition 2.3, there are rational numbers $\alpha_1, \ldots, \alpha_{N+1}$ such that $p(x) = \sum_{i=1}^{N+1} \alpha_i (x+i)^N$. Moreover, α_i has logarithmic size at most $max + poly(N)$.

Therefore we achieve an approximation of $p(x)$ over the interval $[-D, D]$ by computing the function

$$\sum_{i=1}^{N+1} \alpha_i \frac{D+N+1}{\alpha})^N \gamma_N(\frac{\alpha}{D+N+1}(x+i)). \tag{2}$$

The approximation error will be at most

$$(N+1) \cdot 2^{max+poly(n)} \cdot (\frac{D+N+1}{\alpha})^N \cdot \delta \cdot \varepsilon$$

and will be decreased to at most ε, if we set $\delta = 1/((N+1) \cdot 2^{max+poly(n)} \cdot (\frac{D+N+1}{\alpha})^N)$.

The final approximation, combining (1) and (2), has the form

$$\sum_{i=1}^{N+1} \alpha_i (\frac{D+N+1}{\alpha})^N L^N \sum_{j=1}^{s} c_j \gamma(\frac{1}{j \cdot L} \cdot \frac{\alpha}{D+N+1}(x+i)) = \sum_{i=1}^{N+1} \sum_{j=1}^{s} \alpha_{i,j} \gamma(\beta_j \cdot (x+i))$$

for appropriate coefficients $\alpha_{i,j}$ and β_j. Thus we approximate $p(x)$ by a neural network with at most $O(N^2) = O(n^{2k})$ gates, depth 2 and approximation error at most ε. The Lipschitz-bound over $[-D, D]$ follows by inspection, since γ has Lipschitz-bound $O(1)$ over $[-\alpha, \alpha]$. □

3 Equivalence of gate functions

The following gate functions will be able to simulate spline circuits.

Definition 3.1 *Let Γ be a class of gate functions and let $g : [1, \infty] \to \mathcal{R}$ be a function.*
(a) *We say that $g : [1, \infty[\to \mathcal{R}$ is fast converging if and only if*

$$| g(x) - g(x+\varepsilon) | = O(\varepsilon / x^2) \text{ for } x \geq 1, \varepsilon \geq 0,$$

$$0 < \int_0^\infty g(1+u^2) du < \infty \text{ and } | \int_x^\infty g(1+u^2) du | = O(\frac{1}{1+\ln(1+x)}) \text{ for all } x \geq 0.$$

(b) *We say that Γ is powerful if and only if at least one function in Γ is non-trivially smooth and there is a fast converging function g which can be approximated for all s (over the domain $[1, 2^s]$) with error 2^{-s} by a Γ-circuit of constant depth, size $s' = poly(s)$ and Lipschitz-bound $2^{s'}$ over $[1, 2^s]$.*

The Lipschitz-type property of fast convergence can be checked easily for differentiable functions by applying the mean value theorem, i.e., if $g'(x) = O(\frac{1}{x^2})$ for $x \geq 1$, then $|g(x) - g(x+\varepsilon)| = O(\varepsilon/x^2)$ for $x \geq 1$. The integral property is verified, once $|g(x)| = O(\frac{1}{x})$ for $x \geq 1$. The function $x^{-\alpha}$, for $\alpha \geq 1$, is a first example of a fast converging function. The next result provides examples of powerful gate functions,

Lemma 3.1 *The set $\{\gamma\}$ is powerful for any of the following gate functions:*
(a) $\gamma(x) = \exp(x)$,
(b) $\gamma(x) = \ln(x)$,
(c) $\gamma(x) = r(x)$, *where $r(x)$ is a rational function which is not a polynomial,*
(d) $\gamma(x) = x^\alpha$, *provided α is not a natural number,*
(e) $\gamma(x) = \frac{1}{1+e^{-x}}$,

Proof. First observe that all mentioned gate functions are non-trivially smooth, since all of them possess a convergent power series with the required properties over some open interval. Thus it suffices to show that $\{\gamma\}$-circuits are capable of tightly approximating a fast converging function.
(a) Let $\gamma(x) = \exp(x)$. Observe that $\exp(-x)$ is fast converging.
(b) Let $\gamma(x) = \ln(x)$. Observe that $\ln'(x) = \frac{1}{x}$ and hence the fast-converging function $\frac{1}{x}$ can be approximated by the $\{\ln\}$-circuit $\frac{\ln(x+\varepsilon)-\ln(x)}{\varepsilon}$ for sufficiently small ε.
(c) Let $\gamma(x) = \frac{p(x)}{q(x)}$ be a rational function and assume that $q(x) \neq 0$ for $x > L$. Then $r(x+L) = p_1(x+L) + \frac{p_2(x+L)}{q(x+L)}$, where the degree of p_2 is less than the degree of q. We apply Proposition 2.4, to (approximately) compute $p_1(x+L)$ by a $\{r(x)\}$-circuit and then to subtract the approximation of $p_1(x+L)$ from $r(x+L)$. Hence, it suffices to show that $\gamma^*(x) = \frac{p_2(x+L)}{q(x+L)}$ is fast converging.

But $|\frac{p_2(x+L)}{q(x+L)}| = O(\frac{1}{x+L})$ and hence $\gamma^*(x)$ satisfies the integral property of fast convergence. The Lipschitz-type property is satisfied as well, since $(\frac{p_2}{q})' = \frac{p_2' \cdot q - p_2 \cdot q'}{q^2} = O(\frac{1}{x^2})$. Hence $\gamma^*(x)$ is fast converging and $\{r\}$ is powerful.
(d) Let $\gamma(x) = x^\alpha$. Observe that γ is fast converging once $\alpha \leq -1$. We approximate $\gamma'(x) = \alpha \cdot x^{\alpha-1}$ by the differential quotient $\frac{\gamma(x+\varepsilon)-\gamma(x)}{\varepsilon}$. Hence, by iterating this process $k+1$ times (for $k = \lceil \alpha \rceil$), we approximate a fast converging function tightly.
(e) Let $\gamma(x) = \sigma(x)$. Then $\{\sigma\}$ is powerful, since $\sigma(-x)$ is fast converging. □

We now show that powerful gate functions have the approximation power of splines,

Theorem 3.1 *Assume that Γ is powerful. Then splines $\leq \Gamma$.*

Observe that Theorem 1.1 is an immediate consequence of Lemma 2.1 (establishing $\{\gamma\} \leq$ *splines* and the combination of Lemma 3.1 and Theorem 3.1.

Proof of Theorem 3.1. Let $\pi(x)$ be a 1-spline of degree s with $\pi(x) = \begin{cases} p(x) & \text{if } x < 0 \\ q(x) & \text{otherwise,} \end{cases}$ where the coefficients of p and q are rational numbers of logarithmic size at most s. According to Proposition 2.1, it suffices to find a Γ-circuit of depth k', size $(s+1)^{k'}$ and Lipschitz-bound $2^{(s+1)^{k'}}$ over $[-2^{3s+1}, 2^{3s+1}]$ which approximates π over the domain $[-2^{3s+1}, 2^{3s+1}]$ with error at most 2^{-s}. Now assume that the following holds,

Claim 1 *There is a constant k such that for each $s \in \mathcal{N}$ a Γ-circuit C_s of*

$$\text{depth } k, \text{ size } (s+1)^k \text{ and Lipschitz-bound } 2^{(s+1)^k} \text{ over } [-1,1]^n$$

computes a function $t_s : [-1,1] \to \mathcal{R}$ which approximates the binary threshold over the domain $[-1,1] - [-2^{-s^3}, 2^{-s^3}]$ with error at most 2^{-s}. Moreover, $|t_s(x)| \leq 2^{O(s^2)}$ for $x \in [-2^{-s^3}, 2^{-s^3}]$.

Observe that Γ-circuits contain a non-trivially smooth gate function and hence polynomials can be approximated with negative exponential accuracy (Proposition 2.4). Therefore Γ-circuits are also capable of approximately multiplying with negative exponential accuracy, since $x \cdot y = ((x+y)^2 - x^2 - y^2)/2$.

By assumption on p and q, $|p(x) - p(0)|, |q(x) - p(0)| \leq s \cdot 2^s \cdot 2^{3s^2+s} \leq 2^{s+s+3s^2+s} \leq 2^{6s^2}$ for $x \in [-2^{3s+1}, 2^{3s+1}]$. We approximate π by the Γ-circuit $C(\pi, s)$ computing the function

$$\pi_s(x) = p(0) + (q(x) - p(0)) \circ t_{s^3}(\frac{x}{2^{3s+1}}) + (p(x) - p(0)) \circ t_{s^3}(-\frac{x}{2^{3s+1}}),$$

where $\circ$ approximates multiplication over $[-2^{O(s^2)}, 2^{O(s^2)}]^2$ with error at most 2^{-s^2}. The depth of $C(\pi, s)$ will be bounded, its size will be polynomial in s and its Lipschitz-bound (over $[-2^{3s+1}, 2^{3s+1}]$) will be upper-bounded by $2^{poly(s)}$. How good is the approximation of π by π_s?

If $2^{-s^9+3s+1} \leq |x| \leq 2^{3s+1}$, then $t_{s^3}(\frac{x}{2^{3s+1}})$ as well as $t_{s^3}(-\frac{x}{2^{3s+1}})$ approximate the binary threshold with error at most 2^{-s^3}. Hence, if $2^{-s^9+3s+1} \leq x \leq 2^{3s+1}$, taking into account the error of approximate multiplication and observing $p(0) = q(0)$,

$$\begin{aligned} |\pi(x) - \pi_s(x)| &= |q(x) - \pi_s(x)| \\ &= |q(x) - p(0) - (q(x) - p(0)) \circ (1 + t_{s^3}(\frac{x}{2^{3s+1}}) - 1) \\ &\quad -(p(x) - p(0)) \circ t_{s^3}(-\frac{x}{2^{3s+1}})| \end{aligned}$$

$$
\begin{aligned}
&\leq \; |q(x) - p(0) - (q(x) - p(0)) \cdot (1 + t_{s^3}(\frac{x}{2^{3s+1}}) - 1) \\
&\quad -(p(x) - p(0)) \cdot t_{s^3}(-\frac{x}{2^{3s+1}})| + 2 \cdot 2^{-s^2} \\
&\leq \; |(q(x) - p(0)) \cdot (1 - t_{s^3}(\frac{x}{2^{3s+1}}))| + |(p(x) - p(0)) \cdot t_{s^3}(-\frac{x}{2^{3s+1}})| + 2 \cdot 2^{-s^2} \\
&\leq \; 2 \cdot 2^{6s^2 - s^3} + 2 \cdot 2^{-s^2} \leq 2^{-s},
\end{aligned}
$$

provided s is sufficiently large. If $0 \leq x \leq 2^{-s^9+3s+1}$, then $|p(x) - p(0)|, |q(x) - q(0)| \leq s \cdot 2^s \cdot 2^{-s^9+3s+1} = 2^{-\Omega(s^9)}$ and hence

$$
\begin{aligned}
|\pi(x) - \pi_s(x)| \; &\leq \; |q(x) - p(0)| + |q(x) - p(0)| \cdot 2^{O(s^6)} + |p(x) - p(0)| \cdot 2^{O(s^6)} + 2 \cdot 2^{-s^2} \\
&\leq \; 2^{-\Omega(s^9)} + 2^{-\Omega(s^9)} \cdot 2^{O(s^6)} + 2^{-\Omega(s^9)} \cdot 2^{O(s^6)} + 2 \cdot 2^{-s^2} \leq 2^{-s},
\end{aligned}
$$

provided s is sufficiently large. Hence approximation error 2^{-s} is reached for domain $[0, 2^{3s+1}]$ (as well as for $[-2^{3s+1}, 0]$ with an analogous argument). Thus it suffices to prove Claim 1.

Since Γ is powerful, a fast converging function g can be approximated with negative exponential accuracy. We set $h(x) = g(1 + x^2)$. The construction of $t_s(x)$ (for $x \in [-1, 1]$) proceeds as follows.

(1) First x is scaled down by computing $y(x) = \frac{x}{2^s}$. Then the sum

$$
z_s(x) = \sum_{i=0}^{s^{10}} (u_{i+1} - u_i) \cdot h(u_i)
$$

is determined, where $u_i = y(x)a^i$ and $a = (1 + 1/s^6)$. $h(x)$ can be approximated, since we can approximately square.

Thus we try to approximately compute the integral $\alpha = \int_0^\infty h(u)du$ with the help of the geometrically distributed knots u_i. Observe that, assuming a good approximation, $z_s(x)$ tends towards α for positive x, whereas $z(y)$ tends towards $-\alpha$ for negative y. Hence we have implemented a first form of thresholding! But the geometrical distribution helps to achieve only a moderate approximation of α which has to be further tightened:

(2) The approximation of α is improved by first approximately computing

$$
t_s^*(x) = (\frac{\alpha - z_s(x)}{2\alpha})^s,
$$

(using a non-trivially smooth gate function in Γ to approximate x^s)

(3) and then by approximately computing

$$
t_s(x) = (1 - t_s^*(x))^s.
$$

We claim that $t_s(x)$ has the desired properties. We first verify that $z_s(x)$ approximates α for positive x. First we observe that h is continuous, since g is continuous. Therefore,

$$\int_{u_i}^{u_{i+1}} h(u)du = (u_{i+1} - u_i))h(w), \text{ for some } w \text{ with } u_i \leq w \leq u_{i+1}.$$

Next we utilize that g is fast converging. In particular,

$$|h(w) - h(u_i)| = |g(1+w^2) - g(1+u_i^2)| = O(\frac{w^2 - u_i^2}{(1+u_i^2)^2}) = O(\frac{u_{i+1}^2 - u_i^2}{(1+u_i^2)^2}).$$

But $u_{i+1}^2 - u_i^2 = (u_{i+1} - u_i) \cdot (u_{i+1} + u_i)$ and $u_{i+1} + u_i = u_i \cdot (a+1) \leq 3 \cdot u_i \leq 3 \cdot (1+u_i^2)$. Hence,

$$|h(w) - h(u_i)| = O(\frac{u_{i+1} - u_i}{1+u_i^2}).$$

Consequently

$$| \int_{u_i}^{u_{i+1}} h(u)du - (u_{i+1} - u_i)h(u_i) \; |= O(\frac{(u_{i+1} - u_i)^2}{1+u_i^2}).$$

This leads to the estimate

$$\begin{aligned} | \; z_s(x) - \int_0^\infty h(u)du \; | \; &= \; | \int_0^{u_0} h(u)du| + O(\sum_{i=0}^{s^{10}} \frac{(u_{i+1} - u_i)^2}{1+u_i^2}) + | \int_{u_{s^{10}+1}}^\infty h(u)du| \\ &\leq \; | \int_0^{u_0} h(u)du| + O(s^{-2}) + | \int_{u_{s^{10}+1}}^\infty h(u)du|. \end{aligned} \quad (3)$$

This follows, since $u_{i+1} - u_i = u_i(a-1)$ and therefore

$$\sum_{i=0}^{s^{10}} \frac{(u_{i+1} - u_i)^2}{1+u_i^2} = O((a-1)^2 \sum_{i=0}^{s^{10}} \frac{u_i^2}{1+u_i^2}) = (a-1)^2 \cdot O(\sum_{i=0}^{s^{10}} 1) = (a-1)^2 \cdot O(s^{10}) = O(s^{-2}).$$

Furthermore, since g is fast converging,

$$| \int_{u_{s^{10}+1}}^\infty h(u)du| = O(\frac{1}{1+\ln(u_{s^{10}+1}+1)} = O(\frac{1}{1+\ln(u_{s^{10}+1})}. \quad (4)$$

But h is continuous and therefore

$$| \int_0^{u_0} h(u)du| \leq \max_{x \in [0,1]} |h(x)| \cdot u_0 = O(u_0) = O(2^{-s}). \quad (5)$$

In summary, applying (3), (4) and (5)

$$| \; z_s(x) - \int_0^\infty h(u)du \; |= O(\frac{1}{1+\ln(u_{s^{10}+1})} + s^{-2}).$$

Thus $|z_s(x) - \alpha| = O(\frac{1}{s^2})$, provided $u_{s^{10}} \geq 2^{s^2}$.

Now, utilizing that $(1+1/m)^m \geq 2$, we obtain for $i = js^6$,

$$u_i = y(x) \cdot a^i = y(x) \cdot (1 + \frac{1}{s^6})^i \geq y(x) \cdot 2^j = \frac{x}{2^s} \cdot 2^j.$$

Thus, whenever $x \geq 2^{-s^3}$, $u_{s^{10}} \geq \frac{x}{2^s} \cdot 2^{s^4} \geq 2^{-s^3-s} \cdot 2^{s^4} \geq 2^{s^2}$ and $z_s(x)$ approximates α within $O(\frac{1}{s^2})$. For $0 \leq x < 2^{-s^3}$, we have

$$\mid z_s(x) - \int_0^\infty h(u)du \mid = O(\frac{1}{1 + \ln(u_{s^{10}+1}) + 1} + s^{-2}) = O(1).$$

If the argument x is negative, we obtain $z_s(x) = -z_s(-x)$ and hence $-\alpha$ is approximated with the above error bounds.

So far we disregarded that $h(x) = g(1 + x^2)$ can only be approximated. But a sufficiently small negative exponential error is obviously achievable by bounded depth circuits of size polynomial in s.

Let us consider the final two operations, namely $t_s^*(x) = (\frac{\alpha - z_s(x)}{2\alpha})^s$ and $t_s(x) = (1 - t_s^*(x))^s$. First t_s^* maps an approximation of α into a neighborhood of 0 and an approximation of $-\alpha$ into a neighborhood of 1. The powering decreases the distance to 0 "exponentially" and increases the distance to 1 polynomially (by a factor of $O(s)$). In particular, for $2^{-s^3} \leq x \leq 1$, $|\alpha - z_s(x)| = O(s^{-2})$ and

$$|t_s^*(x)| = O((\frac{s^{-2}}{2\alpha})^s) = O(\frac{1}{s^2})^s.$$

For $-1 \leq x \leq -2^{-s^3}$ we obtain $\frac{\alpha - z_s(x)}{2\alpha} = 1 + \varepsilon$, where $|\varepsilon| = O(s^{-2})$. Since $|(1+\varepsilon)^s| = O(1 + |\varepsilon| \cdot s)$ for $|\varepsilon| = O(s^{-2})$,

$$|1 - t_s^*(x)| = |1 - (1+\varepsilon)^s| = O(|\varepsilon| \cdot s) = O(\frac{1}{s}).$$

t_s on the other hand interchanges 0 and 1. Consequently, the previously poor approximation of 1 is turned into a tight approximation of 0, whereas the previously tight approximation of 0 deteriorates, but only polynomially. In particular we get, with the same reasoning as above,

$$\begin{array}{lcll} |1 - t_s(x)| & = & s \cdot O(\frac{1}{s^2})^s & \text{if } 2^{-s^3} \leq x \leq 1 \\ |t_s(x)| & = & O(\frac{1}{s})^s & \text{if } -1 \leq x \leq -2^{-s^3}. \end{array}$$

Thus the result, for $|x| \geq 2^{-s^3}$, is an approximation error of at most 2^{-s} for sufficiently large s. For $|x| < 2^{-s^3}$, we have $|z_s(x)| = O(1)$ and hence $|t_s(x)| = 2^{O(s^2)}$ as claimed. □

Remark 3.1 *Obviously, $1/x$ is powerful. Therefore Theorem 3.1 implies that $\{1/x\}$-circuits of constant depth and size polynomial in s approximate the linear spline spline $\mid x \mid$ over the domain*

$[-1,1]$ *with error at most* 2^{-s}. *Therefore Theorem 3.1 generalizes Newman's approximation of* $|x|$ *by rational functions (Newman,* 1964*), since the degree of the resulting rational function will be polynomial in* s. *(But our approximation requires a polynomially larger degree, when compared with Newman's approximation.)*

Next we consider the consequences of Theorem 3.1 for **binary input**. In particular we would like to compare the computing power of threshold-circuits with the computing power of $\{\sigma\}$-circuits (for $\sigma(x) = \frac{1}{1+e^{-x}}$).

Remark 3.2 **(a)** *Of course, the equivalence of spline circuits and* $\{\sigma\}$*-circuits also holds for binary input. Thus, since threshold circuits can efficiently approximate polynomials and splines (Reif,* 1987*), we obtain that* $\{\sigma\}$*-circuits of depth* d*, size* s *and Lipschitz-bound* 2^s *over* $[-1,1]^n$ *can be simulated by circuits of binary thresholds. The depth of the simulating threshold circuit will increase by a constant factor and its size will increase by a polynomial in* $(s+n)$*, where* n *is the number of input bits. (The inclusion of* n *accounts for the additional increase in size when approximately computing a weighted sum by a threshold circuit.)*

In (DasGupta and Schnitger, 1995) it is shown that the inclusion of n *is also required: a certain family of* n*-bit languages can be computed by* $\{\sigma\}$*-circuits with two gates, whereas threshold circuits of size* $\Omega(\log_2 n)$ *are required.*

(b) *If we allow size to increase by a polynomial in* $s+n$*, then threshold circuits and* $\{\sigma\}$*-circuits have equivalent computing power. This follows from (a), since a threshold gate can be approximately implemented by a sigmoidal gate (Maass et al.,* 1991*).*

(c) *The equivalence of* $\{\sigma\}$*-circuits and threshold circuits does* **not** *hold for analog input, as we will see when considering linear splines.*

Our next goal is to verify that neither degree-bounded polynomials, the *sine*-function nor linear splines are equivalent to σ. We start with polynomials and linear splines.

Proposition 3.1 **(a)** *If a polynomial circuit of size* s *and depth* d *approximates* $|x|$ *over the interval* $[-1,1]$*, then the approximation error will be at least* $s^{-O(d)}$.

(b) *If a circuit of linear 1-splines (of size* s *and depth* d*) approximates* x^2 *over the interval* $[-1,1]$*, then its approximation error will be at least* $s^{-O(d)}$.

Proof. **(a)** Let C be a polynomial circuit (see Definition 1.5) of depth d and size s. Then, assuming that C computes a univariate function, C will compute a polynomial of degree at most s^{d-1}. But a polynomial of degree δ approximates $\mid x \mid$ with error at least $O(1/\delta)$ (page 59 in (Feinerman and Newman, 1974)).

(b) Let C be a circuit of linear 1-splines. Assume that C has size s and depth d. Obviously, C computes a linear spline with at most s^{d-1} knots. Thus there will be an interval $I = [x, x+2\varepsilon]$ of length $2\varepsilon \geq s^{-(d-1)}$ such that C computes a linear function $l : I \to \mathcal{R}$. But observe that

$$l(x) - 2l(x+\varepsilon) + l(x + 2\cdot\varepsilon) = 0$$

whereas

$$x^2 - 2(x+\varepsilon)^2 + (x+2\cdot\varepsilon)^2 = 2\varepsilon^2.$$

And hence C approximates x^2 with error at least $\frac{1}{2}\cdot\varepsilon^2 = s^{-O(d)}$. □

Since both x^2 and $\mid x \mid$ can be computed exactly by degree 2 splines, linear 1-splines and degree-bounded polynomials are weaker than spline circuits. Moreover, linear splines and degree-bounded polynomials are incomparable with respect to error 2^{-s}. Finally, we consider the *sine*-function.

Definition 3.2 *Let $f : [a,b] \to \mathcal{R}$ be a function and let ε be a positive real number. We say that f ε-oscillates t times if and only if there are real numbers $a \leq x_1 < \ldots < x_{t+1} \leq b$ such that*

(a) $f(x_1) = f(x_2) = \ldots = f(x_{t+1})$,

(b) $|x_{i+1} - x_i| \geq \varepsilon$ *for all i and*

(c) *there are real numbers $y_1, \ldots, y_t$ such that $x_i \leq y_i \leq x_{i+1}$ and $|f(x_i) - f(y_i)| \geq \varepsilon$ for all i.*

Part (b) of the next proposition establishes that $\{\sigma\}$- and $\{sine\}$-circuits are not equivalent.

Proposition 3.2 *Assume that $\Gamma \leq$ splines.*

(a) *Let $f : [-1,1] \to \mathcal{R}$ be a function that ε-oscillates t times and let C be a Γ-circuit of depth d, size s and Lipschitz-bound 2^s over $[-1,1]$. If C approximates f with error at most $\frac{\varepsilon}{4}$, then $s \geq t^{\alpha/d}$.*

(The positive constant α depends only on Γ.)

(b) *A Γ-circuit of depth d, which approximates $sine(Ax)$ with error at most $\frac{1}{2}$, has to have size at least $A^{\Omega(1/d)}$.*

(c) *(Bit extraction.) Let $f_A : [-1,1] \to \mathcal{R}$ be any function such that for each integer i $(1 \leq i \leq A)$*

$$f_A(i/A) \;=\; \text{the least significant bit of } i.$$

Then the result of part (b) also applies to f_A.

Proof.

(a) By assumption, C can be approximated (with error 2^{-s}) by a spline-circuit S of size $(s+1)^k$, depth $k(d+1)$, degree s and Lipschitz-bound 2^{s^k} over $[-1,1]$. With Remark 2.1, S will compute a spline of degree at most $s^{k(d+1)}$ and with at most $(s+1)^{k^2(d+1)-1}$ knots. Therefore, S oscillates at most $s^{O(d)}$ times. To achieve t oscillations, its size has to satisfy $s \geq t^{\Omega(1/d)}$. The claim follows by observing that less than t oscillations lead to an error larger than $\varepsilon/4$.

(b) is an immediate consequence of part (a).

(c) Again, the lower bound is an immediate consequence of part (a). □

Remark 3.3 *The lower bound of Proposition 3.2 is "tight" for any class* Γ *which is equivalent to splines. We give an almost matching upper bound: In depth* $O(d)$ *and size* $A^{O(1/d)}$ *compute a binary approximation of the argument* x *with error at most* A^{-d}*. Then utilize that* Γ*-circuits can approximate the binary threshold tightly and determine* $i = \lfloor \frac{A \cdot x}{2\pi} \rfloor$ *(Reif, 1987). Finally approximate* $sine(A \cdot x - i \cdot 2\pi)$ *by tightly approximating the Taylor polynomial. The overall error will be bounded by* $A^{-\Omega(d)}$*.*

4 Conclusions and Open Problems

Our results show that good approximation performance (for error 2^{-s}) depends on efficient approximations of degree s polynomials and on efficient approximations of the binary threshold. Efficient approximations of polynomials succeed for the large class of non-trivially smooth functions. We defined the class of powerful functions which achieve efficient approximations of the binary threshold.

Since (non-polynomial) rational functions are powerful, we were able to generalize Newman's approximation of $|\, x \,|$ by rational functions.

Moreover the standard sigmoid is a powerful function and this frequently utilized gate function reaches an approximation power comparable to or better than classes of more established functions investigated in approximation theory (i.e. splines and polynomials). Additionally, the standard sigmoid is actually more powerful than linear splines, since the standard sigmoid is able to take advantage of its (non-trivial) smoothness to allow more efficient circuits.

The following problems remain open.

- Does *sine* possess the approximation power of the standard sigmoid? Are *sine* and the standard sigmoid incomparable relative to the reducibility $\leq$?

- Does the considerable approximation power of the standard sigmoid translate into a good learning performance?

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