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On theoretical and empirical algorithmic analysis of the efficiency gap measure in partisan gerrymandering

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Abstract Partisan gerrymandering is a major cause for voter disenfranchisement in United States. However, convincing US courts to adopt specific measures to quantify gerrymandering has been of limited success to date. Recently, Stephanopoulos and McGhee in several papers introduced a new measure of partisan gerrymandering via the so-called ”efficiency gap” that computes the absolute difference of wasted votes between two political parties in a two-party system; from a *legal point of view* the measure was found *legally convincing* in a US appeals court in a case that claims that the legislative map of the state of Wisconsin was gerrymandered. The goal of this article is to formalize and provide theoretical and empirical algorithmic analyses of the computational problem of minimizing this measure. To this effect, we show the following:

▷ On the theoretical side, we formalize the corresponding minimization problem and provide non-trivial mathematical and computational complexity properties

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of the problem of minimizing the efficiency gap measure. Specifically, we prove the following results for the formalized minimization problem:

(***i***) We show that the efficiency gap measure attains only a finite discrete set of rational values. (observations of similar nature but using different arguments were also made independently by Cho and Wendy (University of Pennsylvania Law Review, 2017).

(***ii***) We show that, assuming P$\neq$ NP, for general maps and arbitrary numeric electoral data the minimization problem does not admit any polynomial time algorithm with finite approximation ratio. Moreover, we show that the problem still remains NP-complete even if the numeric electoral data is linear in the number of districts, provided the map is provided in the form of a planar graph (or, equivalently, a polygonal subdivision of the two-dimensional Euclidean plane).

(***iii***) Notwithstanding the previous hardness results, we show that efficient exact or efficient approximation algorithms can be designed if one assumes some reasonable restrictions on the map and electoral data.

Items (***ii***) and (***iii***) mentioned above are the first non-trivial computational complexity and algorithmic analyses of this measure of gerrymandering.

▷ On the empirical side, we provide a simple and fast algorithm that can "ungerrymander" the district maps for the states of Texas, Virginia, Wisconsin and Pennsylvania (based on the efficiency gap measure) by bring their efficiency gaps to acceptable levels from the current unacceptable levels. To the best of our knowledge, ours is the first publicly available implementation and its corresponding evaluation on real data for any algorithm for the efficiency gap measure. Our work thus shows that, notwithstanding the general worst-case approximation hardness of the efficiency gap measure as shown by us, finding district maps with acceptable levels of efficiency gaps could be a *computationally tractable problem from a practical point of view*. Based on these empirical results, we also provide some interesting insights into three practical issues related the efficiency gap measure.

Keywords gerrymandering · efficiency gap measure · efficient algorithms · computational complexity

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Mathematics Subject Classification (2010) 68Q17 · 68Q25 · 68W20 · 68W25 · 68W40 · 90C59

1 Introduction and background

Gerrymandering, namely *deliberate* creations of district maps with *highly* asymmetric electoral outcomes to disenfranchise voters, has continued to be a curse to fairness of electoral systems in USA for a long time in spite of general public disdain for it. There is a *long* history of this type of voter disenfranchisement going back as early as 1812 when the specific term "gerrymandering" was coined after a redistricting of the senate election map of the state of Massachusetts resulted in a South Essex district taking a shape that resembled a *salamander* (see Fig. 1). There is an elaborate history of *litigations* involving gerrymandering as well. In 1986 the US Supreme Court (*SCOTUS*) ruled that gerrymandering is *justiciable* [14], but they could *not* agree on an effective way of estimating it. In 2006, *SCOTUS* opined that a measure of *partisan symmetry* may be a helpful tool to understand and remedy gerrymandering [26], but again a precise quantification of partisan symmetry that will be acceptable to the courts was left *undecided*. Indeed, formulating precise and computationally efficient measures for partisan bias (*i.e.*, lack of partisan symmetry) that will be acceptable in *courts* may be considered *critical* to removal of gerrymandering. Partisan symmetry is a standard for defining partisan gerrymandering that involves the computation of counterfactuals typically under the assumption of uniform swings. To illustrate lack of partisan symmetry, consider a two-party voting district and suppose that Party A wins by getting 60% of total votes and 70% of total seats. In such a case, a partisan symmetry standard would hold if Party B would also win 70% of the seats had it won 60% of the votes in a *hypothetical* election[1,2].

Fig. 1: [49] “Gerry” and “salamander” districts, 1812 state senate election, Massachusetts.

[1] Even though measuring partisan bias *is* a non-trivial issue, it has nonetheless been observed that two frequent indicators for partisan bias are *cracking* [33] (dividing supporters of a specific party between two or more districts when they could be a majority in a single district) and *packing* [33] (filling a district with more supporters of a specific party as long as this does not make this specific party the winner in that district). Other partisan bias indicators include *hijacking* [33] (re-districting to force two incumbents to run against each other in one district) and *kidnapping* [33] (moving an incumbent’s home address into another district).

[2] See Section 2.4 regarding the impact of the *SCOTUS* gerrymandering ruling on 06/27/2019 on future gerrymandering studies.

There have been many theoretical and empirical attempts at remedying the lack of partisan symmetry by "quantifying" gerrymandering and devising redistricting methods to optimize such quantifications using well-known notions such as *compactness* and *symmetry* [3, 7–9, 21, 23, 30, 31]. Since it is often simply *not* possible to go over every possible redistricting map to optimize the gerrymandering measure due to rapid *combinatorial explosion*, researchers such as [8, 27, 37] have also investigated designing efficient *algorithmic approach* for this purpose. In particular, a popular gerrymandering measure in the literature is *symmetry*, which attempts to quantify the discrepancy between the share of votes and the share of seats of a party [3, 21, 23, 31]. In spite of such efforts, their success in convincing courts to adopt one or more of these measures has been unfortunately somewhat limited to date.

In recent years, researchers Stephanopoulos and McGhee in two papers [28, 36] have introduced a new gerrymandering measure called the "efficiency gap". Informally speaking, the efficiency gap measure attempts to *minimize* the *absolute difference* of total wasted votes between the parties in a two-party electoral system. This measure is very promising in several aspects. Firstly, it provides a mathematically precise measure of gerrymandering with many desirable properties. Equally importantly, at least from a legal point of view, this measure was found legally convincing in a US appeals court in a case that claims that the legislative map of the state of Wisconsin is gerrymandered.

2 Informal overview of our contribution and its significances

This article is motivated by the following high-level aims:

- *Formalization of the efficiency gap measure and corresponding problem statements*.
- *Computational complexity analysis of these problems*. For those who may wonder why computational complexity analysis (including computational hardness results) may of be practical interest at all, we point out a few reasons.
  - ▷ When a particular type of gerrymandering solution *is* found acceptable in courts, one would eventually need to develop and implement a software for this solution, especially for large US states such as California and Texas where manual calculations may take too long or may not provide the best result. Any exact or approximation algorithms designed by researchers would be a valuable asset in that respect. Conversely, appropriate computational hardness results can be used to convince a court to *not* apply that measure for specific US states due to practical infeasibility.
  - ▷ Beyond scientific implications, we believe computational complexity analysis may also be expected to have a beneficial impact on the US judicial system by showing that the theoretical methods, whether complicated or not (depending on one's background), *can* in fact yield fast accurate computational methods that can be applied to "un-gerrymander" the currently gerrymandered maps.
- *Designing, implementing and evaluating heuristics that work well in practice*.

Redistricting based on minimizing the efficiency gap measure however requires one to find a solution to a combinatorial optimization problem. To this effect, the contribution of this article is as follows:

- ▷ As a necessary first step towards investigating the efficiency gap measure, in Section 3 we first formalize the optimization problem that corresponds to minimizing the efficiency gap measure.
- ▷ Subsequently, in Section 4 we study the mathematical properties of the formalized version of the measure. Specifically, Lemma 1 and Corollary 1 show that the efficiency gap measure attains only a *finite discrete set* of *rational* values; these properties are of considerable importance in understanding the sensitivity of the measure and in designing efficient algorithms for computing this measure[3].
- ▷ Next, in Sections 5 and 6 we investigate computational complexity and algorithm design issues of redistricting based on the efficiency gap measure. Although Theorem 1 and Theorem 2 show that in theory one can construct *artificial pathological examples* for which designing efficient algorithms is provably hard, Theorem 3 and Theorem 4 provide justification as to why the results in Theorem 1 and Theorem 2 are overly pessimistic for real data that do not necessarily correspond to these pathological examples. For example, assuming that the districts are *geometrically compact* (*y-convex* in our terminology), Theorem 4 shows how to find a district map *efficiently in polynomial time* that *minimizes* the efficiency gap. *These are the first non-trivial theoretical computational complexity and algorithmic analyses of the efficiency gap measure*.
- ▷ Finally, to show that it is indeed possible *in practice* to solve the problem of minimization of the efficiency gap, in Section 7 we design a *fast randomized* algorithm based on the *local search paradigm in combinatorial optimization* for this problem (cf. Fig. 7). Our resulting software was tested on four electoral data for the 2012 election of the (federal) house of representatives for the US states of Wisconsin [42, 43], Texas [44, 45], Virginia [40, 41] and Pennsylvania [38, 39]. *The results computed by our fast algorithm are truly outstanding: the final efficiency gap was lowered to* 3.80%, 3.33%, 3.61% *and* 8.64% *from* 14.76%, 4.09%, 22.25% *and* 23.80% *for Wisconsin, Texas, Virginia and Pennsylvania, respectively, in a small amount of time*. Our empirical results clearly show that it is very much possible to design and implement a very fast algorithm that can "un-gerrymander" (*based on the efficiency gap measure*) the gerrymandered US house districts of four US states. *To the best of our knowledge, ours is the first publicly available implementation and its corresponding evaluation on real data for any algorithm for the efficiency gap measure*.

  Based on these empirical results, we also provide some interesting insights into three practical issues related the efficiency gap measure, namely issues pertaining to *seat gain vs. efficiency gap*, *compactness vs. efficiency gap* and *the naturalness of original gerrymandered districts*.

[3]Observations of similar nature but using different arguments were also made independently in [10].

2.1 Implications of results and proofs of Theorem 1 and Theorem 2 in the context of gerrymandering in US

The results in Theorem 1 and Theorem 2 are computational hardness result, so one obvious question is about the implications of these results and associated proofs for gerrymandering in US. To this effect, we offer the following motivations and insights that might be of independent interest.

On accurate census data at the fine granularity level: Accurate census data at the fine granularity level may make a difference to an independent commission seeking fair districts (such as in California). As stated in Remark 1, while it is difficult to even approximately optimize the absolute difference of the wasted votes at a course granularity level of inputs, the situation at the fine granularity level of inputs may be not so hopeless.

On cracking and packing, how far one can push? It is well-known that cracking and packing may result in large partisan bias. For example, based on 2012 election data for election of the (federal) house of representatives for the states of Virginia, the Democratic party had a normalized vote count of about 52% but due to cracking/packing held *only* 4 of the 11 house seats [40, 41]. This observation, coupled with the knowledge that Virginia is one of the *most* gerrymandered states in US both on the congressional and state levels [48], leads to the following natural question: **"could the Virginia lawmakers have disadvantaged the Democratic party more by even more careful execution of cracking and packing approaches"**? As one lawmaker put it quite bluntly, they would have liked to gerrymander more *if only* they could.

We believe a partial answer to this is provided by the proof structures for Theorems 1 and 2. A careful inspection of the proofs of Theorems 1 and 2 reveal that they *do* use cracking and packing[4] to create hard instances of the efficiency gap minimization problem that are computationally intractable to solve optimally certainly at the course granularity input level and *even* at the fine granularity input level[5]. Perhaps the computational complexity issues *did* save the Democratic party from further electoral disadvantages.

2.2 Some remarks and explanations regarding the technical content of this paper

To avoid any possible misgivings or confusions regarding the technical content of the paper as well as to help the reader towards understanding the remaining content of this article, we believe the following comments and explanations may be relevant. *We encourage the reader to read this section and explore the references mentioned therein before proceeding further*.

► We employ a randomized local-search heuristic for combinatorial optimization for our algorithm in Fig. 7. Our algorithmic paradigm is *quite different* from *Markov*

[4] For example, packing is used in the proof of Theorem 2 when a node v_i^3 with 4δ extra supporters for Party **A** is packed in the *same* district with the three nodes $v_{i,p}$, $v_{i,q}$ and $v_{i,r}$ each having δ extra supporters for Party **B** (see Fig. 5).

[5] The proofs of Theorems 1 and 2 however do *not* make much use of *hijacking* or *kidnapping*.

Chain Monte Carlo simulation, *simulated annealing approach*, *Bayesian methods* and related similar other methods (*e.g.*, no temperature parameter, no Gibbs sampling, no calculation of transition probabilities based on Markov chain properties, *etc*.). Thus, for example, our algorithmic paradigm and analysis for the efficiency gap measure is different and incomparable to that used by researchers for other measures, such as by Herschlag, Ravier and Mattingly [24], by Fifield *et al.* [17] or by Cho and Liu [9][6].

For a detailed exposition of *randomized algorithms* the reader is referred to excellent textbooks such as [2, 29] and for a detailed exposition of the *local-search algorithmic paradigm* in combinatorial optimization the reader is referred to the excellent textbook [1].

- ► While we do provide several non-trivial theoretical algorithmic results, we do not provide any theoretical analysis of the randomized algorithm in Fig. 7. The justification for this is that, due to Theorem 1 and Lemma 5, *no such non-trivial theoretical algorithmic complexity results exist* in general assuming P$\neq$ NP for deterministic polynomial-time algorithms or assuming RP$\neq$ NP for randomized polynomial-time algorithms. One can attribute this to the usual "difference between theory and practice" doctrine.

  For readers unfamiliar with the complexity-theoretic assumptions P$\neq$ NP and RP$\neq$ NP, these are *core* complexity-theoretic assumptions that have been routinely used for decades in the field of algorithmic complexity analysis. For example, starting with the famous Cook's theorem [12] in 1971 and Karp's subsequent paper in 1972 [25], the P$\neq$ NP assumption is a central assumption in structural complexity theory and algorithmic complexity analysis. For a detailed technical coverage of the basic structural complexity field, we refer the reader to the excellent textbook [5].
- ► For empirical results in this article we use the data at the county level as opposed to using data at finer (more granular) level such as the "Voting Tabulation District" (VTD) level[7] or the census block level. The reason for this is as follows. Note that our algorithmic approach already returns an efficiency gap of below 4% for three states (namely, WI, TX and VA), and for PA it cuts down the current efficiency gap by a factor of about 3 (*cf.* Table 2). This, together with the observation in [36, pp. 886-888] that the efficiency gap should *not* be minimized to a very low value to avoid unintended consequences, shows that even just by using county-level data our algorithm can already output *almost desirable* (if not truly desirable) values of the efficient gap measure and thus, by *Occam's razor* principle[8] widely used in computer science, we should not be using more data at finer levels. In fact, using more data at a finer level may lead to what is popularly known as "overfitting" in the context of machine learning and elsewhere [6] that may hide its true performance on yet unexplored maps.

[6] In fact, we did try a more traditional simulated annealing approach that is more in tune with some of the previous researchers, but it did not give good results.

[7] VTDs are the smallest units in a state for which the election data are available.

[8] Occam's razor principle [32] states that "*Entia non sunt multiplicanda praeter necessitatem*" (*i.e.*, more things should not be used than are necessary). It is also known as *rule of parsimony* in biological context [18]. Overfitting is an example of *violation* of this principle.

Algorithmic approaches that use data at a level courser than that the most granular level have been used in prior published works, especially when compactness and similar metrics are used to measure and eliminate gerrymandering. For example, Cirincione *et al.* [11] and Doyle [15] use electoral data at a "census block group" level (which is less granular than the data at the "census block" level). In this context, our suggestion to future algorithmic researchers in this direction is to use a minimal amount of data that is truly necessary to generate an acceptable solution.

► In this article we are not comparing our approaches empirically to those in existing literature such as in [9, 17, 24]. The reason for this is that, to the best of our knowledge, there is currently *no* other published work that gives a software to optimize the *efficiency gap measure*. In fact, it would be *grossly unfair to other existing approaches* if we compare our results with their results. For example, suppose we consider an optimal result using an approach from [9] and find that it gives an efficiency gap of 15% whereas the approach in this article gives an efficiency gap of 5%. However, it would be grossly unfair to say that, based on this comparison, our algorithm is better than the one in [9] since the authors in [9] *never* intended to minimize the efficiency gap. Furthermore, even the two maps cannot be compared directly by geometric methods since *no court* has so far established a firm and unequivocal *ground truth* on gerrymandering by having a ruling of the following form:

[court]: "~~a district map is gerrymandered *if and only if* such-and-such conditions are satisfied~~"

(the line is crossed out above just to doubly clarify that such a ruling does not exist).

For certain scientific research problems, algorithmic comparisons are possible because of the existence of ground truths (also called "gold standards" or "benchmarks"). For example, different algorithmic approaches for *reverse engineering* causal relationships between components of a *biological cellular system* can be compared by evaluating how close the methods under investigation are in recovering known gold standard networks using widely agreed upon metrics such as *recall rates* or *precision values* [13]. Unfortunately, for gerrymandering this is not the case and, in our opinion, comparison of algorithms for gerrymandering that optimize *substantially different* objectives should be viewed with a grain of salt.

► *The research goal of this paper is to study minimization of the efficiency gap measure* exactly as introduced by Stephanopoulos and McGhee in *[28, 36] without combining it with any other approaches*. However, should future researchers like to introduce additional computable constraints or objectives, such as compactness or respect of community boundaries, on top of our efficiency gap minimization algorithm, it is a conceptually easy task to modify our algorithm in Fig. 7 for this purpose. For example, to introduce compactness on top of minimization of the efficiency gap measure, the following two lines in Fig. 7

$$\text{if } \mathsf{Effgap}_\kappa(\mathscr{P},\mathscr{Q}'_1,\ldots,\mathscr{Q}'_\kappa) < \mathsf{Effgap}_\kappa(\mathscr{P},\mathscr{Q}_1,\ldots,\mathscr{Q}_\kappa) \text{ then}$$

should be changed to something like (changes are indicated in **bold**):

if $\mathsf{Effgap}_\kappa(\mathscr{P},\mathscr{Q}'_1,\ldots,\mathscr{Q}'_\kappa) < \mathsf{Effgap}_\kappa(\mathscr{P},\mathscr{Q}_1,\ldots,\mathscr{Q}_\kappa)$
and each of $\mathscr{Q}'_1,\ldots,\mathscr{Q}'_\kappa$ are compact then

and appropriate *minor* changes can be made to other parts of the algorithm for consistency with this modification. *However, the computational complexity issues of these new hybrid algorithms may be quite different from the original efficiency gap minimization problem and these issues are not studied in this paper.*

2.3 Beyond scientific curiosity: impact on US judicial system

Beyond its scientific implications on the science of gerrymandering, we expect our algorithmic analysis and results to have a beneficial impact on the US judicial system also. Some justices, whether at the Supreme Court level or in lower courts, seem to have a reluctance to taking mathematics, statistics and computing seriously [16, 35], We sincerely hope that our theoretical and computational results will add the growing body of scientific research on gerrymandering to show that the math, whether complicated or not (depending on one's background), *can* in fact yield fast accurate computational methods that can indeed be applied to un-gerrymander (based on specific quantifiable measures) those maps that are currently gerrymandered.

2.4 Remarks on the impact of the *SCOTUS* gerrymandering ruling

As this article was being written, *SCOTUS* issued a ruling on 06/27/2019 on two gerrymandering cases [34]. However, the ruling does not eliminate the need for future gerrymandering studies. While *SCOTUS* agreed that gerrymandering was anti-democratic, it decided that it is best settled at the legislative and political level, and it encouraged solving the problem at the state court level and delegating legislative redistricting to independent commissions via referendums. Both of the last two remedies do require further scientific studies on gerrymandering. It is also possible that a future *SCOTUS* may overturn this recent ruling.

3 Formalization of minimization of the efficiency gap measure

Based on [28, 36], we abstract our problem in the following manner. We are given a rectilinear polygon $\mathscr{P}$ without holes. Placing $\mathscr{P}$ on a unit grid of size $m \times n$, we will identify an individual unit square (a "*cell*") on the i^{th} row and j^{th} column in $\mathscr{P}$ by $p_{i,j}$ for $0 \leq i < m$ and $0 \leq j < n$ (see Fig. 2). For each cell $p_{i,j} \in \mathscr{P}$, we are given the following three integers:

- ► an integer $\text{Pop}_{i,j} \geq 0$ (the "total population" inside $p_{i,j}$), and
- ► two integers $\text{PartyA}_{i,j}, \text{PartyB}_{i,j} \geq 0$ (the total number of voters for Party A and Party B, respectively) such that $\text{PartyA}_{i,j} + \text{PartyB}_{i,j} = \text{Pop}_{i,j}$.

Let $|\mathscr{P}| = \left|\left\{p_{i,j} \,:\, p_{i,j} \in \mathscr{P}\right\}\right|$ denote the "size" (number of cells) of $\mathscr{P}$. For a rectilinear polygon $\mathscr{Q}$ included in the interior of $\mathscr{P}$ (*i.e.*, a connected subset of the interior of $\mathscr{P}$), we defined the following quantities:

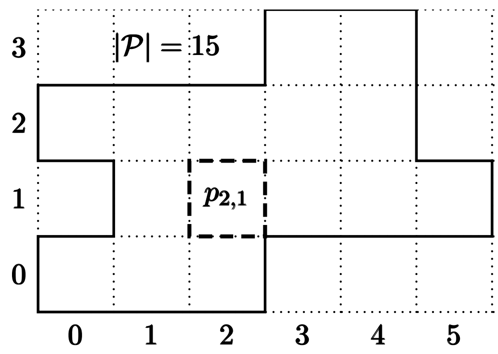


Fig. 2: Input polygon $\mathscr{P}$ of size 15 placed on a grid of size 6×4; the cell $p_{2,1}$ is shown.

Party affiliations in $\mathscr{Q}$: $\text{PartyA}(\mathscr{Q}) = \sum_{p_{i,j} \in \mathscr{Q}} \text{PartyA}_{i,j}$ and $\text{PartyB}(\mathscr{Q}) = \sum_{p_{i,j} \in \mathscr{Q}} \text{PartyB}_{i,j}$.
Population of $\mathscr{Q}$: $\text{Pop}(\mathscr{Q}) = \text{PartyA}(\mathscr{Q}) + \text{PartyB}(\mathscr{Q})$.
Efficiency gap of $\mathscr{Q}$:

$$\mathsf{Effgap}(\mathscr{Q}) = \begin{cases} \Big(\text{PartyA}(\mathscr{Q}) - \frac{1}{2}\text{Pop}(\mathscr{Q})\Big) - \text{PartyB}(\mathscr{Q}) = 2\text{PartyA}(\mathscr{Q}) - \frac{3}{2}\text{Pop}(\mathscr{Q}), \\ \qquad \text{if PartyA}(\mathscr{Q}) \geq \frac{1}{2}\text{Pop}(\mathscr{Q}) \\ \text{PartyA}(\mathscr{Q}) - \Big(\text{PartyB}(\mathscr{Q}) - \frac{1}{2}\text{Pop}(\mathscr{Q})\Big) = 2\text{PartyA}(\mathscr{Q}) - \frac{1}{2}\text{Pop}(\mathscr{Q}), \\ \qquad \text{otherwise} \end{cases}$$

Note that if $\text{PartyA}(\mathscr{Q}) = \text{PartyB}(\mathscr{Q}) = \frac{1}{2}\text{Pop}(\mathscr{Q})$ then $\mathsf{Effgap}(\mathscr{Q}) = -\text{PartyB}(\mathscr{Q})$, *i.e.*, *in case of a tie, we assume Party A is the winner*.

Our problem can now be defined as follows.

Problem name: κ-district Minimum Wasted Vote Problem (MIN-WVP$_\kappa$).

Input: a rectilinear polygon $\mathscr{P}$ with $\text{Pop}_{i,j}, \text{PartyA}_{i,j}, \text{PartyB}_{i,j}$ for every cell $p_{i,j} \in \mathscr{P}$, and a positive integer $1 < \kappa \leq |\mathscr{P}|$.

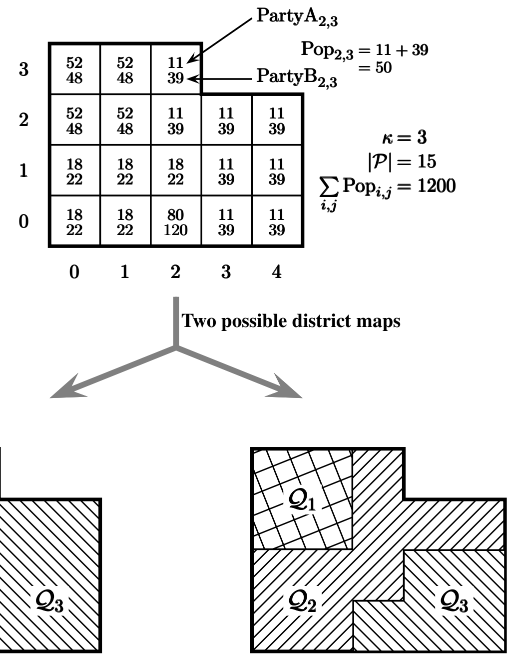
Definition: a κ-equipartition of $\mathscr{P}$ is a partition of the interior of $\mathscr{P}$ into exactly κ rectilinear polygons (*districts*), say $\mathscr{Q}_1, \ldots, \mathscr{Q}_\kappa$, such that $\text{Pop}(\mathscr{Q}_1) = \cdots = \text{Pop}(\mathscr{Q}_\kappa)$.
Assumption: $\mathscr{P}$ has at least one κ-equipartition.

Valid solution: Any κ-equipartition $\mathscr{Q}_1, \ldots, \mathscr{Q}_\kappa$ of $\mathscr{P}$.

Objective: *minimize* the *total* absolute efficiency gap[9]
$\mathsf{Effgap}_\kappa(\mathscr{P}, \mathscr{Q}_1, \ldots, \mathscr{Q}_\kappa) = \left| \sum_{j=1}^{\kappa} \mathsf{Effgap}(\mathscr{Q}_j) \right|$.

Notation: $\mathsf{OPT}_\kappa(\mathscr{P}) \stackrel{\text{def}}{=} \min \big\{ \mathsf{Effgap}_\kappa(\mathscr{P}, \mathscr{Q}_1, \ldots, \mathscr{Q}_\kappa) \mid \mathscr{Q}_1, \ldots, \mathscr{Q}_\kappa \text{ is a } \kappa\text{-equipartition of } \mathscr{P} \big\}$.

Illustration of the formalization: a numerical example To help the reader understand the formalization, we provide a toy numerical example in Fig. 3.



	PartyA($\mathcal{Q}_{...}$)	PartyB($\mathcal{Q}_{...}$)	Effgap($\mathcal{Q}_{...}$)
$\mathcal{Q}_1$	208	192	−184
$\mathcal{Q}_2$	170	230	140
$\mathcal{Q}_3$	88	312	−24
$\mathsf{Effgap}(\mathcal{P}, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3) = \lvert -184 + 140 - 24 \rvert = 68$			

	PartyA($\mathcal{Q}_{...}$)	PartyB($\mathcal{Q}_{...}$)	Effgap($\mathcal{Q}_{...}$)
$\mathcal{Q}_1$	208	192	−184
$\mathcal{Q}_2$	134	266	58
$\mathcal{Q}_3$	124	276	48
$\mathsf{Effgap}(\mathcal{P}, \mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3) = \lvert -184 + 58 + 48 \rvert = 78$			

Fig. 3: A numerical example to illustrate the notations related to the definition of Problem MIN-WVP$_\kappa$ in Section 3.

4 Mathematical properties of efficiency gap: set of attainable values

The following lemma sheds some light on the set of rational numbers that the total efficiency gap of a κ-equipartition can take. As an illustrative example, if we just partition the polygon $\mathscr{P}$ into $\kappa = 2$ regions, then $\mathsf{Effgap}_2(\mathscr{P}, \mathscr{Q}_1, \mathscr{Q}_2)$ can only be

[9] Note that our notation uses the absolute value for $\mathsf{Effgap}_\kappa(\mathscr{P}, \mathscr{Q}_1, \ldots, \mathscr{Q}_\kappa)$ but *not* for individual $\mathsf{Effgap}(\mathscr{Q}_j)$'s.

one of the following 3 possible values:

$$\left|2\,\text{PartyA}(\mathscr{P})-\frac{1}{2}\,\text{Pop}(\mathscr{P})\right| \quad \text{or} \quad \left|\,2\,\text{PartyA}(\mathscr{P})-\text{Pop}(\mathscr{P})\,\right| \quad \text{or} \quad \left|2\,\text{PartyA}(\mathscr{P})-\frac{3}{2}\,\text{Pop}(\mathscr{P})\right|$$

Observations of similar nature but using different arguments were also made independently in [10].

Lemma 1

(*a*) *For any* κ*-equipartition* $\mathscr{Q}_1,\dots,\mathscr{Q}_\kappa$ *of* $\mathscr{P}$*,* $\mathsf{Effgap}_\kappa(\mathscr{P},\mathscr{Q}_1,\dots,\mathscr{Q}_\kappa)$ *always assumes one of the* $\kappa+1$ *values of the form* $\left|2\,\text{PartyA}(\mathscr{P})-\left(z+\frac{\kappa}{2}\right)\frac{\text{Pop}(\mathscr{P})}{\kappa}\right|$ *for* $z=0,1,\dots,\kappa$*.*

(*b*) *If* $\mathsf{Effgap}_\kappa(\mathscr{P},\mathscr{Q}_1,\dots,\mathscr{Q}_\kappa)=\left|2\,\text{PartyA}(\mathscr{P})-\left(z+\frac{\kappa}{2}\right)\frac{\text{Pop}(\mathscr{P})}{\kappa}\right|$ *for some* $z\in\{0,1,\dots,\kappa\}$ *and some* κ*-equipartition* $\mathscr{Q}_1,\dots,\mathscr{Q}_\kappa$ *of* $\mathscr{P}$*, then* $\frac{\text{Pop}(\mathscr{P})}{2\kappa}z\leq\text{PartyA}(\mathscr{P})\leq\frac{\text{Pop}(\mathscr{P})}{2\kappa}z+\frac{1}{2}\text{Pop}(\mathscr{P})$*.*

Corollary 1 *Using the reverse triangle inequality of norms, the absolute difference between two successive values of* $\mathsf{Effgap}_\kappa(\mathscr{P},\mathscr{Q}_1,\dots,\mathscr{Q}_\kappa)$ *is given by*

$$\begin{aligned}&\left|\,\left|2\,\text{PartyA}(\mathscr{P})-\left(\frac{z}{\kappa}-\frac{1}{2}\right)\text{Pop}(\mathscr{P})\right|-\left|2\,\text{PartyA}(\mathscr{P})-\left(\frac{z+1}{\kappa}-\frac{1}{2}\right)\text{Pop}(\mathscr{P})\right|\,\right|\\ &\leq\\ &\left|\left(2\,\text{PartyA}(\mathscr{P})-\left(\frac{z}{\kappa}-\frac{1}{2}\right)\text{Pop}(\mathscr{P})\right)-\left(2\,\text{PartyA}(\mathscr{P})-\left(\frac{z+1}{\kappa}-\frac{1}{2}\right)\text{Pop}(\mathscr{P})\right)\right|\\ &=\frac{\text{Pop}(\mathscr{P})}{\kappa}\end{aligned}$$

Corollary 2 (see also [36, p. 853]) *For any* κ*-equipartition* $\mathscr{Q}_1,\dots,\mathscr{Q}_\kappa$ *of* $\mathscr{P}$*, consider the following quantities as defined in* [36]*:*

(Normalized) seat margin of Party A: $\dfrac{\left|\left\{\mathscr{Q}_j\,:\,\text{PartyA}(\mathscr{Q}_j)\geq\frac{1}{2}\text{Pop}(\mathscr{Q}_j)\right\}\right|}{\kappa}-\dfrac{1}{2}.$

(Normalized) vote margin of Party A: $\dfrac{\text{PartyA}(\mathscr{P})}{\text{Pop}(\mathscr{P})}-\dfrac{1}{2}.$

Then, we can write $\dfrac{2\,\text{PartyA}(\mathscr{P})-\left(z+\frac{\kappa}{2}\right)\frac{\text{Pop}(\mathscr{P})}{\kappa}}{\text{Pop}(\mathscr{P})}$ *as* $2\left(\dfrac{\text{PartyA}(\mathscr{P})}{\text{Pop}(\mathscr{P})}-\dfrac{1}{2}\right)-\left(\dfrac{z}{\kappa}-\dfrac{1}{2}\right)$, *and identifying z with the quantity* $\left|\left\{\mathscr{Q}_j\,:\,\text{PartyA}(\mathscr{Q}_j)\geq\frac{1}{2}\text{Pop}(\mathscr{Q}_j)\right\}\right|$ *we get*

$$\frac{\mathsf{Effgap}_\kappa(\mathscr{P},\mathscr{Q}_1,\dots,\mathscr{Q}_\kappa)}{\text{Pop}(\mathscr{P})}=\left|2\times(\text{ vote margin of Party A })-(\text{ seat margin of Party A })\right|$$

Proof of Lemma 1

(a) Consider any κ-equipartition $\mathscr{Q}_1,\dots,\mathscr{Q}_\kappa$ of $\mathscr{P}$ with $\text{Pop}(\mathscr{Q}_1)=\dots\text{Pop}(\mathscr{Q}_\kappa)=\frac{1}{\kappa}\text{Pop}(\mathscr{P})$. Note that for any $\mathscr{Q}_j$ we have $\mathsf{Effgap}(\mathscr{Q}_j)=2\text{PartyA}(\mathscr{Q}_j)-r_j\text{Pop}(\mathscr{Q})$ where

$$r_j=\begin{cases} 3/2, \text{ if PartyA}(\mathscr{Q}_j)\geq \text{Pop}(\mathscr{Q}_j)/(2\,\kappa)\\ 1/2, \text{ otherwise}\end{cases}$$

Letting z be the number of r_j's that are equal to $3/2$, it follows that

$$\begin{aligned}\mathsf{Effgap}_\kappa(\mathscr{P},\mathscr{Q}_1,\dots,\mathscr{Q}_\kappa)&=\left|\sum_{j=1}^{\kappa}\mathsf{Effgap}(\mathscr{Q}_j)\right|\\ &=\left|2\text{PartyA}(\mathscr{P})-\left(\frac{3}{2}z+\frac{1}{2}(\kappa-z)\right)\frac{\text{Pop}(\mathscr{P})}{\kappa}\right|\\ &=\left|2\,\text{PartyA}(\mathscr{P})-\left(z+\frac{\kappa}{2}\right)\frac{\text{Pop}(\mathscr{P})}{\kappa}\right|\end{aligned}$$

(b) Note that, since $0\leq \text{PartyA}(\mathscr{Q}_j)\leq\frac{\text{Pop}(\mathscr{P})}{\kappa}$ for any j, we have

$$\text{PartyA}(\mathscr{P})=\sum_{j=1}^{\kappa}\text{PartyA}(\mathscr{Q}_j)\geq\sum_{j\,:\,r_j=3/2}\frac{\text{Pop}(\mathscr{P})}{2\,\kappa}=\frac{\text{Pop}(\mathscr{P})}{2\,\kappa}z$$

$$\begin{aligned}\text{PartyA}(\mathscr{P})=\sum_{j=1}^{\kappa}\text{PartyA}(\mathscr{Q}_j)&<\sum_{j\,:\,r_j=3/2}\frac{\text{Pop}(\mathscr{P})}{\kappa}+\sum_{j\,:\,r_j=1/2}\frac{\text{Pop}(\mathscr{P})}{2\,\kappa}\\ &=\frac{\text{Pop}(\mathscr{P})}{\kappa}z+\frac{\text{Pop}(\mathscr{P})}{2\,\kappa}(\kappa-z)=\frac{\text{Pop}(\mathscr{P})}{2\,\kappa}z+\frac{1}{2}\text{Pop}(\mathscr{P})\quad\square\end{aligned}$$

5 Computational hardness result for MIN-WVP$_\kappa$

Recall that, for any $\rho\geq 1$, an approximation algorithm with an approximation ratio of ρ (or, simply an ρ-approximation) is a polynomial-time solution of value at most ρ times the value of an optimal solution [19].

Theorem 1 *Assuming* $\mathsf{P}\neq\mathsf{NP}$, *for any rational constant* $\varepsilon\in(0,1)$, *the* MIN-WVP$_\kappa$ *problem for a rectilinear polygon* $\mathscr{P}$ *does not admit a* ρ*-approximation algorithm for any* ρ *and all* $2\leq\kappa\leq\varepsilon|\mathscr{P}|$.

Proof. We reduce from the NP-complete PARTITION problem [19] which is defined as follows: *given a set of n positive integers* $\mathscr{A}=\{a_0,\dots,a_{n-1}\}$, *decide if there exists a subset* $\mathscr{A}'\subset\mathscr{A}$ *such that* $\sum_{a_i\in\mathscr{A}'}a_i=\sum_{a_j\notin\mathscr{A}'}a_j$. Note that we can assume without loss of generality that n is sufficiently large and each of $a_0,\dots,a_{n-1}$ is a multiple of *any* fixed positive integer. For notational convenience, let $\Delta=\sum_{j=0}^{n-1}a_j$.

Let $\mu\geq 0$ be such that $\kappa=2+\mu n$ (we will later show that μ is at most the constant $\frac{6\varepsilon}{1-\varepsilon}$). Our rectilinear polygon $\mathscr{P}$, as illustrated in Fig. 4 (A), consists of a rectangle $\mathscr{C}=\left\{p_{i,j}\,|\,0\leq i\leq n,\,0\leq j\leq 2\right\}$ of size $3\times(n+1)$ with additional μn cells

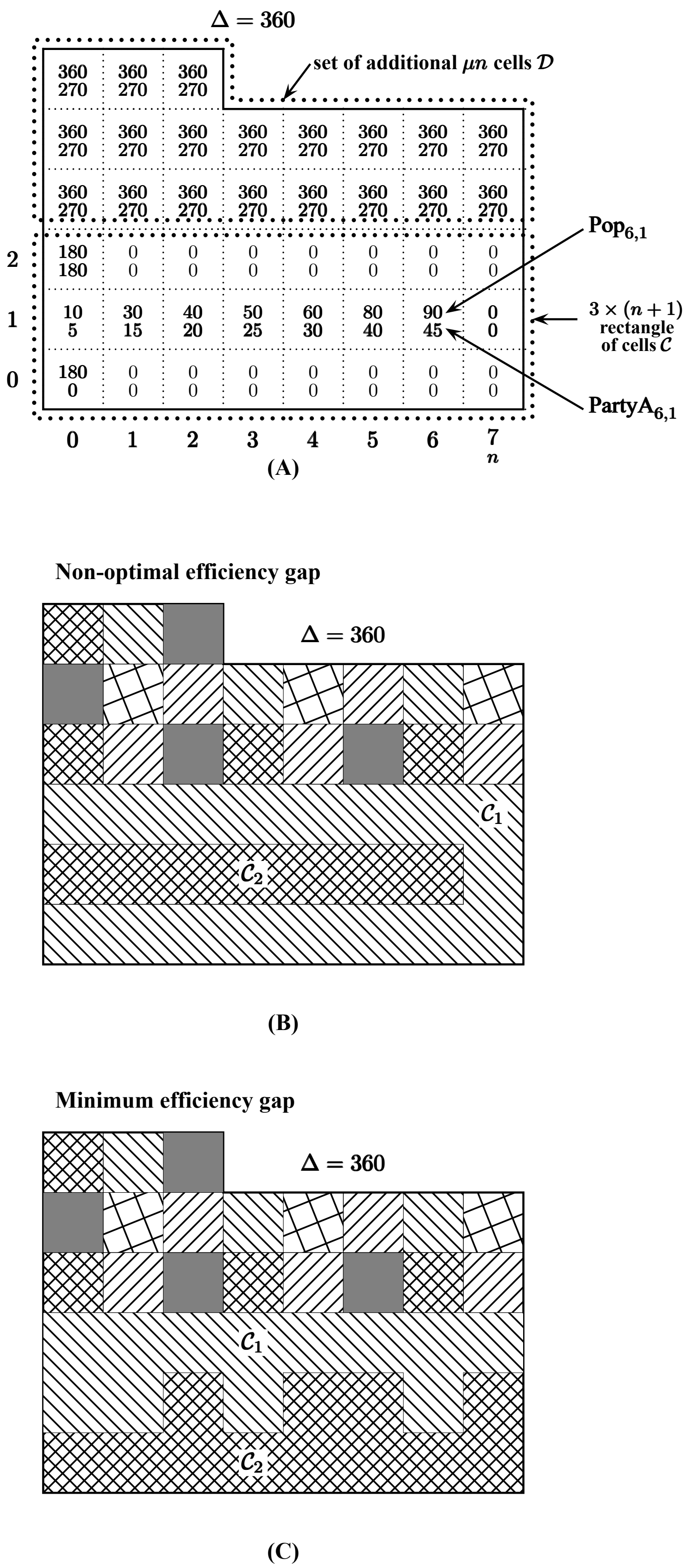


Fig. 4: An illustration of the construction in the proof of Theorem 1 when the instance of the PARTITION problem is $\mathscr{A} = \{10, 30, 40, 50, 60, 80, 90\}$. **(A)** The instance of MIN-WVP$_\kappa$ created from the given instance of PARTITION. **(B)** Efficiency gap when an invalid solution of PARTITION is used. **(C)** Efficiency gap when a valid solution of PARTITION exists and is used.

attached to it in any arbitrary manner to make the whole figure a connected polygon without holes. For convenience, let $\mathscr{D} = \{p_{i,j} \,|\, p_{i,j} \notin \mathscr{C}\}$ be the set of the additional μn cells. The relevant numbers for each cell are as follows:

$$\text{Pop}_{i,j} = \begin{cases} \Delta/2, \text{ if } i = j = 0 \text{ or if } i = 0, j = 2 \\ a_j, \text{ if } i = 1 \text{ and } j < n \\ \Delta, \text{ if } p_{i,j} \in \mathscr{D} \\ 0, \text{ otherwise} \end{cases} \qquad \text{PartyA}_{i,j} = \begin{cases} \Delta/2, \text{ if } i = j = 0 \\ a_j/2, \text{ if } i = 1 \text{ and } j < n \\ 3\Delta/4, \text{ if } p_{i,j} \in \mathscr{D} \\ 0, \text{ otherwise} \end{cases}$$

First, we show how to select a rational constant μ such that any integer κ in the range $[2, \varepsilon|\mathscr{P}|]$ can be realized. Assume that $\kappa = \varepsilon'|\mathscr{P}| \in [2, \varepsilon|\mathscr{P}|]$ for some ε'. Since $|\mathscr{P}| = 3(n+1) + \mu n$ the following calculations hold:

$$\kappa = 2 + \mu n = \varepsilon'|\mathscr{P}| = \varepsilon'(3(n+1) + \mu n) \equiv \mu = \frac{3\varepsilon'(n+1) - 2}{(1-\varepsilon')n} < \frac{4\varepsilon'}{1-\varepsilon'} < \frac{4\varepsilon}{1-\varepsilon}$$

Claim 1.1 $\mathsf{Effgap}(p_{i,j}) = 0$ for each $p_{i,j} \in \mathscr{D}$, and moreover each $p_{i,j} \in \mathscr{D}$ must be a separate partition by itself in any κ-equipartition of $\mathscr{P}$.

Proof. By straightforward calculation, $\mathsf{Effgap}(p_{i,j}) = 2 \times \frac{3\Delta}{4} - \frac{3\Delta}{2} = 0$. Since $\kappa = 2 + \mu n$ and $\text{Pop}(\mathscr{P}) = \sum_{p_{i,j} \in \mathscr{P}} \text{Pop}_{i,j} = \Delta + \Delta + \mu n \Delta = (2 + \mu n)\Delta$, each partition in any κ-equipartition of $\mathscr{P}$ must have a population of $\frac{\text{Pop}(\mathscr{P})}{\kappa} = \Delta$ and thus each $p_{i,j} \in \mathscr{D}$ of population Δ must be a separate partition by itself. ❑

Using Claim 1.1 we can simply ignore all $p_{i,j} \in \mathscr{D}$ in the calculation of of efficiency gap of a valid solution of $\mathscr{P}$ and it follows that the total efficiency gap of a κ-equipartition of $\mathscr{P}$ is identical to that of a 2-equipartition of $\mathscr{C}$. A proof of the theorem then follows provided we prove the following two claims.

(soundness) If the PARTITION problem does not have a solution then $\mathsf{OPT}_2(\mathscr{C}) = \Delta$.

(completeness) If the PARTITION problem has a solution then $\mathsf{OPT}_2(\mathscr{C}) = 0$.

Proof of soundness (refer to Fig. 4 (B))

Suppose that there exists a valid solution (*i.e.*, a 2-equipartition) $\mathscr{C}_1, \mathscr{C}_2$ of MIN-WVP$_2$ for $\mathscr{C}$ with $p_{0,0} \in \mathscr{C}_1, p_{0,2} \in \mathscr{C}_2$, and let $\mathscr{A}' = \{a_j \,|\, p_{1,j} \in \mathscr{C}_1\}$. Then,

$$\Delta = \frac{\text{Pop}(\mathscr{C})}{2} = \text{Pop}_{0,0} + \sum_{p_{1,j} \in \mathscr{C}_1} \text{Pop}_{1,j} = \frac{\Delta}{2} + \sum_{a_j \in \mathscr{A}'} a_j \equiv \sum_{a_j \in \mathscr{A}'} a_j = \frac{\Delta}{2}$$

and thus $\mathscr{A}'$ is a valid solution of PARTITION, a contradiction!

Thus, assume that both $p_{0,0}$ and $p_{0,2}$ belong to the same partition, say $\mathscr{C}_1$. Then, since $\text{Pop}_{0,0} + \text{Pop}_{0,4} = \Delta = \frac{\text{Pop}(\mathscr{C})}{2}$, every $p_{1,j}$ must belong to $\mathscr{C}_2$. Moreover, every $p_{i,j} \in \mathscr{C}$ with $\eta_{i,j} = 0$ must belong to $\mathscr{C}_1$ since otherwise $\mathscr{C}_1$ will *not* be a connected region. This provides $\text{Pop}(\mathscr{C}_1) = \text{Pop}(\mathscr{C}_2) = \Delta$, showing that $\mathscr{C}_1, \mathscr{C}_2$ is indeed a valid solution (*i.e.*, a 2-equipartition) of MIN-WVP$_2$ for $\mathscr{C}$. The total efficiency gap of this solution can be calculated as

$$\begin{aligned}\mathsf{Effgap}_2(\mathscr{C},\mathscr{C}_1,\mathscr{C}_2) &= |\,\mathsf{Effgap}(\mathscr{C}_1)+\mathsf{Effgap}(\mathscr{C}_2)\,| \\ &= \left|2\,\mathrm{PartyA}(\mathscr{C}_1)-\frac{3}{2}\mathrm{Pop}(\mathscr{C}_1)+2\,\mathrm{PartyA}(\mathscr{C}_1)-\frac{3}{2}\mathrm{Pop}(\mathscr{C}_1)\right| \\ &= \left|2\frac{\Delta}{2}-\frac{3}{2}\Delta+2\frac{\Delta}{2}-\frac{3}{2}\Delta\right| = \Delta\end{aligned}$$

Proof of completeness (refer to Fig. 4 (C))

Suppose that there is a valid solution of $\mathscr{A}' \subset \mathscr{A}$ of PARTITION and consider the two polygons

$$\mathscr{C}_1 = \left\{p_{2,j}\,|\,0 \le j \le n\right\} \cup \left\{p_{1,j}\,|\,a_j \in \mathscr{A}'\right\}, \quad \mathscr{C}_2 = \mathscr{C} \setminus \mathscr{C}_1$$

By straightforward calculation, it is easy to verify the following:

- $\mathrm{Pop}(\mathscr{C}_1) = \sum_{a_j \in \mathscr{A}'} a_j + \sum_{j=0}^{n} \mathrm{Pop}_{2,j} = \Delta$, $\mathrm{Pop}(\mathscr{C}_2) = \sum_{a_j \notin \mathscr{A}'} a_j + \sum_{j=0}^{n} \mathrm{Pop}_{2,j} = \Delta$, and thus $\mathscr{C}_1,\mathscr{C}_2$ is a valid solution (*i.e.*, a 2-equipartition) of MIN-WVP$_2$ for $\mathscr{C}$.
- $\mathsf{Effgap}_2(\mathscr{C},\mathscr{C}_1,\mathscr{C}_2) = 0\mathsf{OPT}_2(\mathscr{C}) = 0$ since

$$\begin{aligned}\mathsf{Effgap}_2(\mathscr{C},\mathscr{C}_1,\mathscr{C}_2) &= |\,\mathsf{Effgap}(\mathscr{C}_1)+\mathsf{Effgap}(\mathscr{C}_2)\,| \\ &= \left|2\,\mathrm{PartyA}(\mathscr{C}_1)-\frac{3}{2}\mathrm{Pop}(\mathscr{C}_1)+2\,\mathrm{PartyA}(\mathscr{C}_1)-\frac{1}{2}\mathrm{Pop}(\mathscr{C}_1)\right| \\ &= \left|2\left(\frac{\Delta}{2}+\frac{\Delta}{4}\right)-\frac{3}{2}\Delta+2\frac{\Delta}{4}-\frac{1}{2}\Delta\right| = 0\end{aligned}$$

❏

5.1 Computational hardness of MIN-WVP$_\kappa$ for polynomial-size total population

Since the PARTITION problem is not a strongly NP-complete problem (*i.e.*, admits a pseudo-polynomial time solution), the approximation-hardness result in Theorem 1 does not hold if the total population $\mathrm{Pop}(\mathscr{P})$ is polynomial in $|\mathscr{P}|$ (*i.e.*, if $\mathrm{Pop}(\mathscr{P}) = O(|\mathscr{P}|^c)$ for some positive constant c). Indeed, if $\mathrm{Pop}(\mathscr{P})$ is polynomial in $|\mathscr{P}|$ then it is easy to design a polynomial-time exact solution via dynamic programming for those instances of MIN-WVP$_\kappa$ problem that appear in the proof of Theorem 1. Thus, an obvious research question is whether one can design an alternate hardness proof that holds even if $\mathrm{Pop}(\mathscr{P})$ is polynomial in $|\mathscr{P}|$. While we are unable to show it for the case when the input is a rectilinear polygon as used to define MIN-WVP$_\kappa$, we can nonetheless show this if the input is a planar graph instead and if we *relax* the definition of a κ-equipartition *slightly*. The relevant notations and terminologies for the planar graph input model and the relaxation of the definition of a κ-equipartition are described as follows[10]:

[10] Alternatively, one can think of the input being given as an *arbitrary* (not necessarily rectilinear) simple polygon $\mathscr{P}$, and the cells are arbitrary sub-polygons (without holes) inside $\mathscr{P}$ that form a partition of $\mathscr{P}$.

- ▷ Instead of the rectilinear polygon $\mathscr{P}$, the input is now a *planar* graph $G = (V,E)$ whose nodes are the cells, and whose edges define adjacency of pairs of cells.
- ▷ The previously used notations and terminologies are modified in the following obvious manner:
  - ▷ $\text{Pop}_{i,j}$ is now replaced by Pop_v (the "total population" in v) for a node (cell) $v \in V$. Similarly, $\text{PartyA}_{i,j}$ and $\text{PartyB}_{i,j}$ are replaced by the notations PartyA_v and PartyB_v, respectively.
  - ▷ For a subset of cells (nodes) $\mathscr{Q} \subseteq V$, $\text{PartyA}(\mathscr{Q})$ is redefined as $\text{PartyA}(\mathscr{Q}) = \sum_{v \in \mathscr{Q}} \text{PartyA}_v$, and $\text{PartyB}(\mathscr{Q})$ is redefined as $\text{PartyB}(\mathscr{Q}) = \sum_{v \in \mathscr{Q}} \text{PartyB}_v$.
  - ▷ $|\mathscr{P}|$ is now $|V|$.
- ▷ A *ε-relaxed κ*-equipartition of G[11], for a given $\varepsilon > 0$, is a partition of V into κ subsets of nodes (κ districts), say $\mathscr{Q}_1, \ldots, \mathscr{Q}_\kappa$, such that:
  - ▷ every $\mathscr{Q}_j$ induces a *connected* subgraph of G, and
  - ▷ $\dfrac{\max_{1 \le j \le \kappa} \{\text{Pop}(\mathscr{Q}_j)\}}{\min_{1 \le j \le \kappa} \{\text{Pop}(\mathscr{Q}_j)\}} \le 1 + \varepsilon.$

Theorem 2 *Computing an exact solution of the* MIN-WVP_κ *problem for planar graph input using ε-approximate κ-equipartitions is* NP*-complete for any constant* $0 < \varepsilon < 1/2$ *even if* $\text{Pop}(V) = O(|V|)$.

Remark 1 The NP-hardness reduction in Theorem 2 does not provide any non-trivial inapproximability ratio.

Proof of Theorem 2 The problem is trivially in NP, so will concentrate on the NP-hardness reduction. Our reduction is from the *maximum independent set problem for planar cubic graphs* (MIS_{PC}) which is defined as follows:

> *"given a cubic (i.e.,* 3*-regular) planar graph* $G = (V,E)$ *and an integer* ν*, does there exist an independent set for* G *with* ν *nodes ?"*

MIS_{PC} is known to be NP-complete [20] but there exists a polynomial-time approximation scheme for it [4]. Note the value of $\text{Effgap}_\kappa(G, \mathscr{Q}_1, \ldots, \mathscr{Q}_\kappa)$ remains the *same* if we divide (or multiply) the values of all $\text{PartyA}(\mathscr{Q}_j)$'s and $\text{PartyB}(\mathscr{Q}_j)$'s by t for any integer $t > 0$. Thus, to simplify notation, we assume that we have re-scaled the numbers such that $\min_{1 \le j \le \kappa} \{\text{Pop}(\mathscr{Q}_j)\} = 1$, and therefore our approximately strict partitioning criteria is satisfied by ensuring that $1 \le \text{Pop}(\mathscr{Q}_j) \le 1 + \varepsilon$ for all $j = 1, \ldots, \kappa$ with $\text{Pop}(\mathscr{Q}_j) = 1$ for at least one j. Thus, each $\text{PartyA}(\mathscr{Q}_j)$, $\text{PartyB}(\mathscr{Q}_j)$ and $\text{Pop}(\mathscr{Q}_j)$ may be positive *rational* constant numbers such that, if needed, we can ensure that all these numbers are integers at the end of the reduction by multiplying them by a suitable positive integer of polynomial size.

Let $G = (V,E)$ and ν be the given instance of MIS_{PC} with $V = \{v_1, \ldots, v_n\}$ and $|E| = 3n/2$. Note that, since G is cubic, we can always greedily find an independent set of at least $n/4$ nodes and moreover there does not exist any independent set of more than $n/2$ nodes; thus we can assume $n/4 < \nu \le n/2$. Let $\delta = n^{-3}/100 > 0$ be a rational

[11] Past Court rulings seem to suggest that the courts may allow a maximum value of ε in the range of 0.05 to 0.1 (*cf.* US Supreme Court ruling in Karcher v. Daggett 1983).

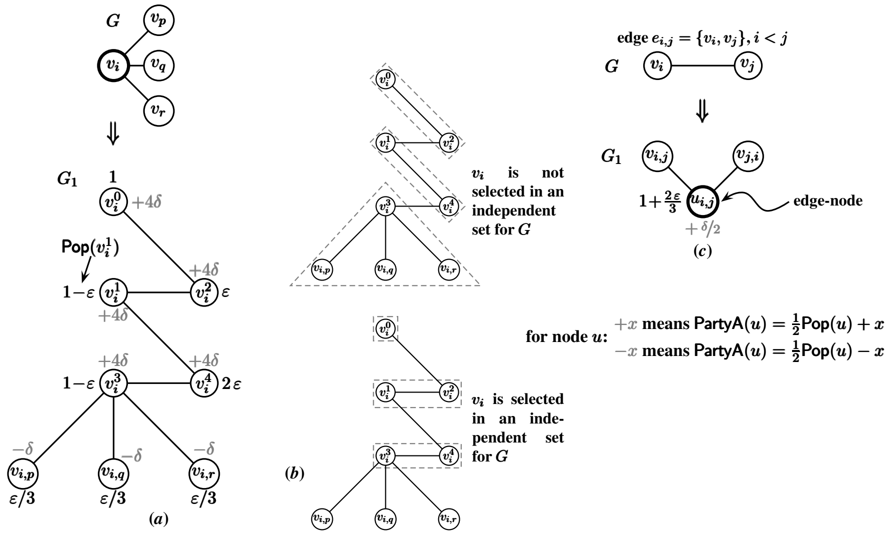


Fig. 5: The subgraph gadgets used in the proof of Theorem 2.

number of polynomial size that is sufficiently small compared to ε. We describe an instance of our map $G_1 = (V_1, E_1)$ (a planar graph with all required numbers) constructed from G as follows.

Node gadgets: Every node $v_i \in V$ with its three adjacent nodes as v_p, v_q, v_r is replaced a subgraph of 8 new nodes $v_i^0, v_i^1, v_i^2, v_i^3, v_i^4, v_{i,p}, v_{i,q}, v_{i,r} \in V_1$ and 7 new edges along with their Pop and PartyA values as shown in Fig. 3 (a). The requirement "$1 \leq \text{Pop}(\mathscr{Q}_j) \leq 1+\varepsilon$ for all j" and the fact that $0 < \varepsilon < 1/2$ ensure that these nodes can be covered only in the two possible ways as shown in Fig. 3 (b):

- ▷ For the top case in Fig. 3 (b), all the 8 nodes are covered by 3 districts. Intuitively, this corresponds to the case when v_i is *not* selected in an independent set for G. We informally refer to this as the *the "v_i is not selected" case*.
- ▷ For the bottom case in Fig. 3 (b), 5 of the 8 nodes are covered by 3 districts, leaving the remaining 3 nodes (nodes $v_{i,p}, v_{i,q}, v_{i,r}$) to be covered with some other nodes in G_1. Intuitively, this corresponds to the case when v_i *is* selected in an independent set for G. We informally refer to this as the *the "v_i is selected" case*.

Note that this step in all introduces $8n$ new nodes and $7n$ new edges in G_1.

Edge gadgets: For every edge $e_{i,j} = \{v_i, v_j\} \in E$ (with $i < j$), we introduce one new node (the "edge-node") $u_{i,j}$ and two new edges $\{v_{i,j}, u_{i,j}\}$ and $\{v_{j,i}, u_{i,j}\}$ as shown in Fig. 3 (c). Note that this step in all introduces $3n/2$ new nodes and $3n$ new edges in G_1.

Thus, we have $|V_1| = 19n/2$ and $|E_1| = 10n$, and surely G_1 is planar since G was a planar graph. Finally, we set $\kappa = 9n/2$. Note that the instance G_1 satisfies $\text{Pop}(V) = O(|V|)$ since the total population of every node is between $\varepsilon/3$ and $1 + (2\varepsilon/3)$ for a constant ε.

To continue with the proof, we need to make a sequence of observations about the constructed graph G_1 as follows:

(i) An edge-node $u_{i,j}$ can be in a partition just by itself, or with only one of either of the nodes $v_{i,j}$ and $v_{j,i}$.
(ii) If v_i is not selected then $u_{i,j}$ cannot be in the same partition as $v_{i,j}$. On the other hand, if $u_{i,j}$ is in the same partition as $v_{i,j}$ then v_i must be selected.
(iii) By ***(i)*** and ***(ii)***, an edge-node $u_{i,j}$ is in a partition just by itself if and only if neither of its end-points, namely nodes v_i and v_j, are selected in the corresponding independent set for G.
(iv) Consider any maximal independent set $\emptyset \subset V' \subset V$ for G (*e.g.*, the one obtained by the obvious greedy solution) having $0 < \mu < n/2$ nodes. Using ***(i)***, ***(ii)*** and ***(iii)***, the following calculations hold:
- ▷ For every node v_i selected in V' with its adjacent nodes being v_p, v_q, v_r, we cover the nodes v_i^0, v_i^1, v_i^2, v_i^3, v_i^4, $v_{i,p}$, $v_{i,q}$, $v_{i,r}$, and the three edge-nodes corresponding to the three edges $\{v_i, v_p\}$, $\{v_i, v_q\}$, $\{v_i, v_r\} \in E$ using 6 districts in G_1.
- ▷ For every node v_i *not* selected in V', we cover the nodes v_i^0, v_i^1, v_i^2, v_i^3, v_i^4, $v_{i,p}$, $v_{i,q}$, and $v_{i,r}$ using 3 districts in G_1.
- ▷ Let $E' \subseteq E$ be the set of edges such that *neither* end-points of these edges are selected in V'. Note that $|E'| = (3n/2) - 3\mu$, and for every edge $v_{i,j} \in E'$ we use one new district for the edge-node $u_{i,j}$.

Lemma 2 *There is a trivial (not necessarily optimal) valid solution for G_1.*

Proof. By ***(iv)***, the total number of districts used in a maximal independent set is $6\mu + 3(n-\mu) + ((3n/2) - 3\mu) = 9n/2 = \kappa$, as required. ❑

For convenience of calculations, let us define the following quantity for a partition (district) for a district $\mathcal{Q}_j$:

$$\text{Wasted-Votes}(\mathcal{Q}_j, \text{Party } \mathbf{A}) = \begin{cases} \text{PartyA}(\mathcal{Q}_j) - \left(\frac{\text{Pop}(\mathcal{Q}_j)}{2}\right), \text{ if PartyA}(\mathcal{Q}_j) \geq \frac{\text{Pop}(\mathcal{Q}_j)}{2} \\ \text{PartyA}(\mathcal{Q}_j), \text{ otherwise} \end{cases}$$

$$\text{Wasted-Votes}(\mathcal{Q}_j, \text{Party } \mathbf{B}) = \begin{cases} \text{PartyB}(\mathcal{Q}_j) - \left(\frac{\text{Pop}(\mathcal{Q}_j)}{2}\right), \text{ if PartyB}(\mathcal{Q}_j) \geq \frac{\text{Pop}(\mathcal{Q}_j)}{2} \\ \text{PartyB}(\mathcal{Q}_j), \text{ otherwise} \end{cases}$$

Then, the following calculations follow easily (for any sufficiently small positive rational number x):

$$\text{Wasted-Votes}(\mathcal{Q}_j, \text{Party } \mathbf{A}) = \begin{cases} x, \text{ if PartyA}(\mathcal{Q}_j) = \frac{\text{Pop}(\mathcal{Q}_j)}{2} + x \\ \frac{\text{Pop}(\mathcal{Q}_j)}{2} - x, \text{ if PartyA}(\mathcal{Q}_j) = \frac{\text{Pop}(\mathcal{Q}_j)}{2} - x \end{cases}$$

$$\text{Wasted-Votes}(\mathcal{Q}_j, \text{Party } \mathbf{B}) = \begin{cases} \frac{\text{Pop}(\mathcal{Q}_j)}{2} - x, \text{ if PartyA}(\mathcal{Q}_j) = \frac{\text{Pop}(\mathcal{Q}_j}{2} + x \\ x, \text{ if PartyA}(\mathcal{Q}_j) = \frac{\text{Pop}(\mathcal{Q}_j)}{2} - x \end{cases}$$

$$\text{Wasted-Votes}(\mathcal{Q}_j, \text{Party } \mathbf{A}) - \text{Wasted-Votes}(\mathcal{Q}_j, \text{Party } \mathbf{B})$$

$$= \begin{cases} 2x - \frac{\text{Pop}(\mathscr{Q}_j)}{2}, \text{ if PartyA}(\mathscr{Q}_j) = \frac{\text{Pop}(\mathscr{Q}_j)}{2} + x \\ \frac{\text{Pop}(\mathscr{Q}_j)}{2} - 2x, \text{ if PartyA}(\mathscr{Q}_j) = \frac{\text{Pop}(\mathscr{Q}_j)}{2} - x \end{cases}$$

Consider any maximal independent set $\emptyset \subset V' \subset V$ for G having $n/4 < \mu \leq n/2$ nodes. Using ***(iv)***, the following calculations hold:

- ▷ Every node v_i selected in V' contributes the following amount to the total value of $\sum_{j=1}^{\kappa} (\mathsf{Wasted\text{-}Votes}(\mathscr{Q}_j, \text{Party } \mathbf{A}) - \mathsf{Wasted\text{-}Votes}(\mathscr{Q}_j, \text{Party } \mathbf{B}))$:

$$\xi = \left(8\delta - \frac{1}{2}\right) + \left(16\delta - \frac{1}{2}\right) + \left(16\delta - \frac{1+\varepsilon}{2}\right) + 3 \times \left(\frac{1}{2} - 2\delta\right) = 34\delta - \frac{\varepsilon}{2}$$

- ▷ Every node v_i *not* selected in V' contributes the following amount to the total value of $\sum_{j=1}^{\kappa} (\mathsf{Wasted\text{-}Votes}(\mathscr{Q}_j, \text{Party } \mathbf{A}) - \mathsf{Wasted\text{-}Votes}(\mathscr{Q}_j, \text{Party } \mathbf{B}))$:

$$\zeta = \left(16\delta - \frac{1+\varepsilon}{2}\right) + \left(16\delta - \frac{1+\varepsilon}{2}\right) + \left(2\delta - \frac{1}{2}\right) = 34\delta - \varepsilon - \frac{3}{2}$$

- ▷ Every edge in E such that *neither* end-points of the edge are selected in V' contributes the following amount to the total value of $\sum_{j=1}^{\kappa} (\mathsf{Wasted\text{-}Votes}(\mathscr{Q}_j, \text{Party } \mathbf{A}) - \mathsf{Wasted\text{-}Votes}(\mathscr{Q}_j, \text{Party } \mathbf{B}))$:

$$\eta = \delta - \frac{1 + \frac{2\varepsilon}{3}}{2} = \delta - \frac{\varepsilon}{3} - \frac{1}{2}$$

- ▷ Consequently, corresponding to an independent set of μ nodes, adding all the contributions we get the following value $\Upsilon(\mu)$ for $\sum_{j=1}^{\kappa} (\mathsf{Wasted\text{-}Votes}(\mathscr{Q}_j, \text{Party } \mathbf{A}) - \mathsf{Wasted\text{-}Votes}(\mathscr{Q}_j, \text{Party } \mathbf{B}))$

$$\begin{aligned} \Upsilon(\mu) &= \mu\xi + (n-\mu)\zeta + \left(\frac{3n}{2} - 3\mu\right)\eta \\ &= \left(34\mu\delta - \frac{\mu\varepsilon}{2}\right) + (n-\mu)\left(34\delta - \varepsilon - \frac{3}{2}\right) + \left(\frac{3n}{2} - 3\mu\right)\left(\delta - \frac{\varepsilon}{3} - \frac{1}{2}\right) \\ &= 3\mu + \left(\frac{3\varepsilon}{2} - 3\delta\right)\mu + \left(\frac{71\delta}{2} - \frac{3\varepsilon}{2} - \frac{9}{4}\right)n \end{aligned}$$

Now we note the following properties of the quantity $\Upsilon(\mu)$:

- ▷ Since $\delta = n^{-3}/100$ and $n/4 < \mu \leq n/2$, we have $\Upsilon(\mu) < 0$ and therefore $|\Upsilon(\mu)| = -\Upsilon(\mu)$.
- ▷ Consequently, $|\Upsilon(\mu)| - |\Upsilon(\mu - 1)| = \Upsilon(\mu - 1) - \Upsilon(\mu) = -3 - \frac{3\varepsilon}{2} + 3\delta$.

The last equality then leads to the following two statements that complete the proof for NP-hardness:

- ► If G has an independent set of ν nodes then $\mathsf{Effgap}_\kappa(G, \mathscr{Q}_1, \ldots, \mathscr{Q}_\kappa) = |\Upsilon(\nu)|$.
- ► If every independent set of G has at most $\nu - 1$ nodes then $\mathsf{Effgap}_\kappa(G, \mathscr{Q}_1, \ldots, \mathscr{Q}_\kappa) \geq |\Upsilon(\nu - 1)| > |\Upsilon(\nu)| + 2$. ❑

6 Efficient algorithms for special cases

Although Theorem 1 seems to render the problem MIN-WVP$_\kappa$ intractable in theory, our empirical results show that the problem is computationally tractable in practice. This is because in real-life applications, many constraints in the theoretical formulation of MIN-WVP$_\kappa$ are often relaxed. For example:

- ***(i)* Restricting district shapes:** Individual partitions of the κ-equipartition of $\mathscr{P}$ may be restricted in shape. For example, 37 states require their legislative districts to be reasonably compact and 18 states require congressional districts to be compact [46].
- ***(ii)* Variations in district populations:** A partition $\mathscr{Q}_1,\ldots,\mathscr{Q}_\kappa$ of $\mathscr{P}$ is only *approximately* κ-equipartition, *i.e.*, $\text{Pop}(\mathscr{Q}_1),\ldots,\text{Pop}(\mathscr{Q}_\kappa)$ are approximately, but not exactly, equal to $\text{Pop}(\mathscr{P})/\kappa$. For example, the usual federal standards require equal population *as nearly as is practicable* for congressional districts but allow more relaxed *substantially equal* population (*e.g.*, no more than 10% deviation between the largest and smallest district) for state and local legislative districts [46].
- ***(iii)* Bounding the efficiency gap measure away from zero:** A κ-equipartition $\mathscr{Q}_1$, $\ldots$, $\mathscr{Q}_\kappa$ of $\mathscr{P}$ is a valid solution only if $\mathsf{Effgap}_\kappa(\mathscr{P},\mathscr{Q}_1,\ldots,\mathscr{Q}_\kappa) \geq \varepsilon\text{Pop}(\mathscr{P})$ for some $0 < \varepsilon < 1$. Indeed, the authors that originally proposed the efficiency gap measure provided in [36, pp. 886-887] several reasons for not requiring the quantity $\mathsf{Effgap}_\kappa(\mathscr{P},\mathscr{Q}_1,\ldots,\mathscr{Q}_\kappa)/\text{Pop}(\mathscr{P})$ to be either zero or too close to zero.

In this section, we explore algorithmic implications of these types of relaxations of constraints for MIN-WVP$_\kappa$.

6.1 The case of two stable and approximately equal partitions

This case considers constraints (*ii*) and (*iii*). The following definition of "near partitions" formalizes the concept of variations in district populations.

Definition 1 (Near partitions) Let $\kappa \in \mathbb{N}$, and let $\mathscr{P}$ be an instance of MIN-WVP$_\kappa$. Let $\phi \stackrel{\text{def}}{=} \mathscr{Q}_i,\ldots,\mathscr{Q}_\kappa$ be such that $\mathscr{Q}_1,\ldots,\mathscr{Q}_\kappa$ is a partition of $\mathscr{P}$, such that for each $i \in \{1,\ldots,\kappa\}$, we have

$$\left(\frac{1}{\kappa}-\delta\right)\text{Pop}(\mathscr{P}) \leq \text{Pop}(\mathscr{Q}_i) \leq \left(\frac{1}{\kappa}+\delta\right)\text{Pop}(\mathscr{P})$$

for some $\delta \geq 0$. Then we say that ϕ is a δ-*near partition*.

The next definition of "stability" formalizes the concept of bounding away from zero the efficiency gap of each partition.

Definition 2 (Stability) Let $\kappa \in \mathbb{N}$, let $\mathscr{P}$ be an instance of MIN-WVP$_\kappa$, and let $\phi \stackrel{\text{def}}{=} \mathscr{Q}_1,\ldots,\mathscr{Q}_\kappa$ be a partition for $\mathscr{P}$. We say that ϕ is γ-*stable*, for some $\gamma > 0$, if for all $i \in \{1,\ldots,\kappa\}$, we have

$$\mathsf{Effgap}(\mathscr{Q}_i) > \gamma\text{Pop}(\mathscr{Q}_i)$$

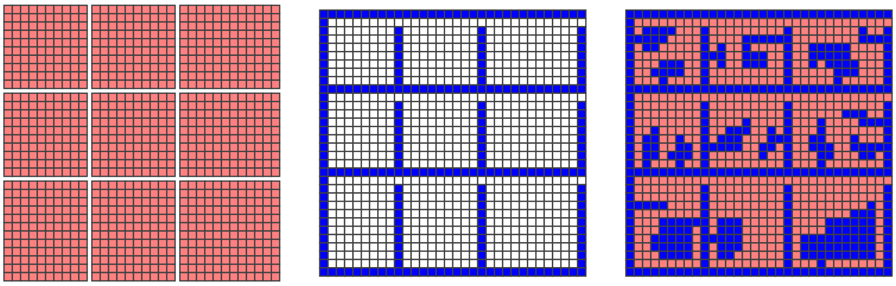


Fig. 6: The 10-basic rectangles of a 32×32 grid (left), the 10-basic tree (middle), and a 10-canonical solution (right).

Theorem 3 *Let $\gamma > 0$ and let $\mathscr{P}$ be an instance of* MIN-WVP$_2$. *Suppose that there are no cells of zero population and the maximum number of people in any cell of $\mathscr{P}$ is C. Suppose that there exists a γ-stable δ-near partition $\phi \stackrel{\text{def}}{=} \mathscr{Q}_1, \mathscr{Q}_2$ of P. Then, for any fixed $\varepsilon > 0$, there exists an algorithm which given $\mathscr{P}$ computes some δ-near partition $\phi' \stackrel{\text{def}}{=} \mathscr{Q}'_1, \mathscr{Q}'_2$ of $\mathscr{P}$, for some $\delta = O(\varepsilon C)$, such that $\mathsf{Effgap}(\phi') \leq (1+\varepsilon)\mathsf{Effgap}(\phi)$, in $2^{O(1/\varepsilon^2)} 2^{O(C/\delta)^2} (nmC)^{O(1)}$ time.*

Proof.

Definition 3 (Canonical solution) Let $\mathscr{P}$ be an instance of MIN-WVP$_2$ and $t \in \mathbb{N}$. Let $\mathscr{B}$ be the partition of $\mathscr{P}$ obtained as follows: We partition $\mathscr{P}$ into $\lfloor m/t \rfloor \cdot \lfloor n/t \rfloor$ rectangles, where each rectangle consists of the intersection of t rows with t columns, except possibly for the rectangles that are incident to the right-most and the bottom-most boundaries of $\mathscr{P}$. We refer to the rectangles in $\mathscr{B}$ as *t-basic* (see Fig. 6). Let T be the set of cells consisting of the union of the left-most column of $\mathscr{P}$, the top row of $\mathscr{P}$, and for each t-basic rectangle X, the bottom row of X, and the right-most column of X, except for the cell that is next to the top cell of that column (see Fig. 6). We refer to T as the *t-basic tree* For each t-basic rectangle X, we define its *interior* to be the set of cells in X that are at distance at least 2 from T. A solution $\phi \stackrel{\text{def}}{=} \mathscr{Z}_1, \mathscr{Z}_2$ of $\mathscr{P}$ is called *t-canonical* if it satisfies the following properties.

(1) $T \subseteq Z_1$.
(2) Let X be a t-basic rectangle, and let X' be its interior. For each $i \in \{1,2\}$, let A_i be the set of connected components of $X' \cap \mathscr{Z}_i$. Then, for each $Y \in A_1$, there exists a unique cell in T_1 that is adjacent to both Y and T. Moreover, all other cells in $X \setminus (X' \cup T)$ are in $\mathscr{Z}_2$ (see Fig. 6).

Lemma 3 *Let $\mathscr{P}$ be an instance of* MIN-WVP$_2$. *Suppose that there are no empty cells and the maximum number of people in any cell of $\mathscr{P}$ is C. Let $\gamma > 0$, and suppose that there exists a γ-stable solution $\phi \stackrel{\text{def}}{=} \mathscr{Q}_1, \mathscr{Q}_2$ for P. Then, for any $\varepsilon > 0$, at least one of the following conditions hold:*

(1) Either $\mathscr{Q}_1$ or $\mathscr{Q}_2$ is contained in some $1/\varepsilon$-basic rectangle.

(2) There exists a $\lceil 1/\varepsilon \rceil$*-canonical* δ*-near solution* $\phi' = \mathscr{Q}'_1, \mathscr{Q}'_2$ *of* $\mathscr{P}$*, for some* $\delta = O(\varepsilon C)$*, such that for all* $i \in \{1,2\}$*, we have* $|\text{PartyA}(\mathscr{Q}_i) - \text{PartyA}(\mathscr{Q}'_i)| \leq O(C\varepsilon n + m)$*, and* $|\text{PartyB}(\mathscr{Q}_i) - \text{PartyB}(\mathscr{Q}'_i)| \leq O(C\varepsilon nm)$*.*

Proof. It suffices to show that if condition (1) does not hold, then condition (2) does. We define a partition $\phi' \stackrel{\text{def}}{=} \mathscr{Q}'_1, \mathscr{Q}'_2$ of $\mathscr{P}$ as follows. We initialize $\mathscr{Q}'_1$ to be empty. Let T be the $1/\varepsilon$-basic tree. We add T_1 to $\mathscr{Q}'_1$. For each $1/\varepsilon$-basic rectangle X, let X' be its interior. For each $i \in \{1,2\}$, let A_i be the set of connected components of $X' \cap \mathscr{Q}_1$. Since ϕ is a valid solution, we have that $\mathscr{Q}_1$ is connected. Since condition (1) does not hold, it follows that $\mathscr{Q}_1$ intersects at least two $1/\varepsilon$-basic rectangles. Therefore, each component $W \in A_1$ must contain some cell c_W on the boundary of X'. By construction, c_W must be incident to some cell c'_W that is incident to T. We add c'_W to $\mathscr{Q}'_1$. Repeating this process for all basic rectangles, and for all components W as above. Finally, we define $\mathscr{Q}'_2 = \mathscr{P} \setminus \mathscr{Q}'_1$. This completes the definition of the partition $\phi' \stackrel{\text{def}}{=} \mathscr{Q}'_1, \mathscr{Q}'_2$ of $\mathscr{P}$. It remains to show that this is the desired solution.

First, we need to show that ϕ' is a valid solution. To that end, it suffices to show that both $\mathscr{Q}'_1$ and $\mathscr{Q}'_2$ are connected. The fact that $\mathscr{Q}'_1$ is connected follows directly from its construction. To show that $\mathscr{Q}'_2$ is connected we proceed by induction on the construction of $\mathscr{Q}'_1$. Initially, $\mathscr{Q}'_1$ consists of just the cells in T, and thus its complement is clearly connected. When we consider a component W, we add $W \cup \{c'_W\}$ to $\mathscr{Q}'_1$. Since we add only a single cell that is incident to both W and $\mathscr{Q}'_1$, it follows inductively that $\mathscr{Q}'_1$ remains simply connected (that is, it does not contain any holes), and therefore its complement remains connected. This concludes the proof that both $\mathscr{Q}'_1$ and $\mathscr{Q}'_2$ are connected, and therefore ϕ' is a valid solution.

The solutions ϕ and ϕ' can disagree only on cells that are not in the interior of any basic rectangle. All these cells are contained in the union of $O(\varepsilon m)$ rows and (εn) columns. Thus, the total number of voters in these cells is at most $O(C\varepsilon nm)$. It follows that for each $i \in \{1,2\}$, we have $|\text{PartyA}(\mathscr{Q}_i) - \text{PartyA}(\mathscr{Q}'_i)| \leq O(C\varepsilon nm)$, and $|\text{PartyB}(\mathscr{Q}_i) - \text{PartyB}(\mathscr{Q}'_i)| \leq O(C\varepsilon nm)$.

Since there are no empty cells, we have $\text{Pop}(\mathscr{P}) \geq nm$. It follows that for all $i \in \{1,2\}$, we have

$$|\text{Pop}(\mathscr{Q}_i) - \text{Pop}(\mathscr{Q}'_i)| \leq O(C\varepsilon nm) \leq C \cdot \varepsilon \cdot \text{Pop}(\mathscr{P}).$$

Thus ϕ' is δ-near, for some $\delta = O(\varepsilon C)$, which concludes the proof. ❑

Lemma 4 *Let* $\gamma > 0$*. Let* $\mathscr{P}$ *be an instance of* MIN-WVP$_2$*. Suppose that there are no empty cells and the maximum number of people in any cell of* $\mathscr{P}$ *is C. Suppose that there exists a* δ*-stable partition* $\phi \stackrel{\text{def}}{=} \mathscr{Q}_1, \mathscr{Q}_2$ *of P. Then, for any fixed* $\varepsilon > 0$*, there exists an algorithm which given* $\mathscr{P}$ *computes some* δ*-near partition* $\phi' \stackrel{\text{def}}{=} \mathscr{Q}'_1, \mathscr{Q}'_2$ *of* $\mathscr{P}$*, for some* $\delta = O(\varepsilon C)$*, such that for all* $i \in \{1,2\}$*, we have* $|\text{PartyA}(\mathscr{Q}_i) - \text{PartyA}(\mathscr{Q}'_i)| \leq O(C\varepsilon nm)$*, and* $|\text{PartyB}(\mathscr{Q}_i) - \text{PartyB}(\mathscr{Q}'_i)| \leq O(C\varepsilon nm)$*, in* $2^{O(1/\varepsilon^2)}(nmC)^{O(1)}$ *time.*

Proof. We can check whether there exist a partition $\phi \stackrel{\text{def}}{=} \mathscr{Q}_1, \mathscr{Q}_2$ satisfying the conditions, and such that either $\mathscr{Q}_1$ or $\mathscr{Q}_1$ is contained in the interior of a single $1/\varepsilon$-basic rectangle. This can be done by trying all $1/\varepsilon$-basic rectangles, and all possible subsets of the interior of each $1/\varepsilon$-basic rectangle, in time $(n/\varepsilon)(m/\varepsilon)2^{O(1/\varepsilon^2)} = nm2^{O(1/\varepsilon^2)}$.

It remains to consider the case where neither of $\mathscr{Q}_1$ and $\mathscr{Q}_2$ is contained in the interior of any $1/\varepsilon$-basic rectangle. It follows that condition (2) of Lemma 3 holds. That is, there exists some $\lceil 1/\varepsilon \rceil$-canonical δ-near solution $\phi' \stackrel{\text{def}}{=} \mathscr{Q}'_1, \mathscr{Q}'_2$ of $\mathscr{P}$, for some $\delta = O(\varepsilon C)$, such that for all $i \in \{1,2\}$, we have $|\text{PartyA}(\mathscr{Q}_i) - \text{PartyA}(\mathscr{Q}'_i)| \leq O(C\varepsilon n + m)$, and $|\text{PartyB}(\mathscr{Q}_i) - \text{PartyB}(\mathscr{Q}'_i)| \leq O(C\varepsilon nm)$. We can compute such a partition ϕ' via dynamic programming, as follows. Let I be the union of the interiors of all $1/\varepsilon$-basic rectangles. By the definition of a canonical partition, it suffices to compute $\mathscr{Q}'_1 \cap I$ and $\mathscr{Q}'_2 \cap I$. Since $\mathscr{Q}'_2 \cap I = I \setminus (\mathscr{Q}'_1 \cap I)$, it suffices to compute $\mathscr{Q}'_1 \cap I$. Let $L_{\text{PartyA}} = \text{PartyA}(\mathscr{Q}'_1 \cap I)$, and $L_{\text{PartyB}} = \text{PartyB}(\mathscr{Q}'_1 \cap I)$. Clearly, $L_{\text{PartyA}}, L_{\text{PartyB}} \in \{0, \ldots, Cnm\}$. Thus there are at most $O((nmC)^2)$ different values for the pair $(L_{\text{PartyA}}, L_{\text{PartyB}})$. We construct a dynamic programming table, containing one entry for each possible value for the pair $(L_{\text{PartyA}}, L_{\text{PartyB}})$. Initially, all entries of the table are unmarked, except for the entry that corresponds to the pair $(0,0)$. We iteratively consider all $1/\varepsilon$-basic rectangles. When considering some $1/\varepsilon$-basic rectangle X, with interior X', we enumerate all possibilities for $Y = X' \cap \mathscr{Q}'_1$. There are $2^{O(1/\varepsilon^2)}$ possibilities for Y. For each such possibility, we update the dynamic programming table by marking the position $(i + \text{PartyA}(Y), j + \text{PartyB}(Y))$, if the position (i, j) is already marked from the previous iteration. The total running time is $2^{O(1/\varepsilon^2)}(nmC)^{O(1)}$. ❑

The proof of Theorem 3 now follows from Lemma 4 by setting $\varepsilon = \min\{\varepsilon, O(\delta/C)\}$. ❑

6.2 The case of convex shaped partitions

This case encompasses constraint (*i*) since convexity has been used in gerrymandering studies such as [47] as a measure of compactness to examine how redistricting reshapes the geography of congressional districts. We recall that some $X \subseteq \mathbb{R}^2$ is called *y-convex* if for every vertical line ℓ, we have that $X \cap \ell$ is either empty, or a line segment. We also say that a κ-partition $\mathscr{Q}_1, \ldots, Q_\kappa$ of $\mathscr{P}$ is *y-convex* if for all $i \in \{1, \ldots, \kappa\}$, $\mathscr{Q}_i$ is *y-convex*.

Theorem 4 *Let $\mathscr{P}$ be a rectilinear polygon realized in the $m \times n$ grid, and let $N =$ Pop$(\mathscr{P})$ be the total population on $\mathscr{P}$. There exists an algorithm for computing a y-convex κ-equipartition of $\mathscr{P}$ of minimum efficiency gap, with running time $N^{O(\kappa)}$. In particular, the running time is polynomial when the total population is polynomial and the total number of partitions is a constant.*

Proof. Let $\mathscr{P}^* \stackrel{\text{def}}{=} \mathscr{Q}^*_1, \ldots, \mathscr{Q}^*_\kappa$ be a y-convex κ-equipartition of $\mathscr{P}$ of minimum efficiency gap. For any $i \in \{1, \ldots, n\}$, let C_i be the i-th column of $\mathscr{P}$. We observe that for all $i \in \{1, \ldots, n\}$, and for all $j \in \{1, \ldots, \kappa\}$, we have that $\mathscr{Q}^*_j \cap C_i$ is either empty, or consists of a single rectangle of width 1. Let $\mathscr{C}_i$ be the set of all partitions of C_i into exactly κ (possibly empty) segments, each labeled with a unique integer in $\{1, \ldots, \kappa\}$.

We further define

$$a^*_{i,j} = \text{PartyA}(\mathscr{Q}^*_j \cap (C_1 \cup \ldots \cup C_i)),$$

and

$$b^*_{i,j} = \text{PartyB}(\mathscr{Q}^*_j \cap (C_1 \cup \ldots \cup C_i)).$$

The algorithm proceeds via dynamic programming. For each $i \in \{1, \ldots, n\}$, let $I_i = \mathbb{N}^{2\kappa} \times \mathscr{S}_i$. Let $X_i = (a_{i,1}, b_{i,1}, \ldots, a_{i,\kappa}, b_{i,\kappa}, \mathscr{Z}_i) \in I_i$. If $i = 1$, then we say that X_i is *feasible* if for all $j \in \{1, \ldots, \kappa\}$, the unique set $Y \in \mathscr{Z}_1$ labeled j satisfies

$$a_{1,j} = \text{PartyA}(Y) \text{ and } b_{1,j} = \text{PartyB}(Y).$$

Otherwise, if $i > 1$, we say that X_i is feasible if the following holds: There exists some feasible $X_{i-1} = (a_{i-1,1}, b_{i-1,1}, \ldots, a_{i-1,\kappa}, b_{i-1,\kappa}, \mathscr{Z}_{i-1}) \in I_{i-1}$, such that for all $j \in \{1, \ldots, \kappa\}$, we have that the unique set $Y \in \mathscr{Z}_i$ labeled j satisfies

$$a_{i,j} = a_{i-1,j} + \text{PartyA}(Y) \text{ and } b_{i,j} = b_{i-1,j} + \text{PartyB}(Y).$$

For each $i \in \{1, \ldots, n\}$ we inductively compute the set $\mathscr{F}_i$ of all feasible $X_i \in I_i$. This can clearly be done in time $N^{O(\kappa)}$. It is immediate that for all $i \in \{1, \ldots, n\}$, there exists some $X_i \in I_i$ that achieves efficiency gap equal to the restriction of $\mathscr{P}^*$ on the union of the first i columns. Thus, by induction on i, the algorithm computes a feasible solution with optimal efficiency gap. ❑

7 Empirical results for real data of four gerrymandered states

In this section, we design a *fast randomized* algorithm based on the *local search paradigm* for the problem of minimization of the efficiency gap measure, and empirically evaluate the algorithm on real data of four gerrymandered states to show that it may be possible *in practice* to effectively minimize the efficiency gap. Our algorithm starts with a given $\mathscr{Q}_1, \ldots, \mathscr{Q}_\kappa$ partition of the input state $\mathscr{P}$. Note that $\mathscr{Q}_1, \ldots, \mathscr{Q}_\kappa$ was only an *approximate* equipartition in the sense that the values $\text{Pop}(\mathscr{Q}_1)$, ..., $\text{Pop}(\mathscr{Q}_\kappa)$ are as close to each other as practically possible but need not be *exactly* equal (*cf.* US Supreme Court ruling in Karcher v. Daggett 1983). For designing alternate valid district plans, we therefore allow any partition $\mathscr{Q}'_1, \ldots, \mathscr{Q}'_\kappa$ of $\mathscr{P}$ such that, for every j, $\min_{1 \le i \le \kappa} \{\text{Pop}(\mathscr{Q}_i)\} \le \text{Pop}(\mathscr{Q}'_j) \le \max_{1 \le i \le \kappa} \{\text{Pop}(\mathscr{Q}_i)\}$.

7.1 Input preprocessing

We preprocess the input map to generate an undirected unweighted planar graph $\mathfrak{G} = (\mathfrak{V}, \mathfrak{E})$. Each node in the graph corresponds to a planar subdivision of a county that is assigned to a district (or to an entire county if it is assigned to a district as a whole). Two nodes are connected by an edge if and only if they share a border on the map. Each node $v \in \mathfrak{V}$ has three corresponding numbers: $\text{PartyA}(v)$ (total number of voters for Party A), $\text{PartyB}(v) =$ (total number of voters for Party B), and $\text{Pop}(v) =$

$\text{PartyA}(v) + \text{PartyB}(v)$ (total population in v)[12]. A district $\mathscr{D}$ is then a connected subgraph of $\mathfrak{G}$ with $\text{PartyA}(\mathscr{D}) = \sum_{v \in \mathscr{D}} \text{PartyA}(v)$ and $\text{PartyB}(\mathscr{D}) = \sum_{v \in \mathscr{D}} \text{PartyB}(v)$.

7.2 Availability and format of raw data

Link to all data files for the three states used in the paper are available in `http://www.cs.uic.edu/~dasgupta/gerrymander/index.html`. Each data is an EXCEL spreadsheet. Explanations of various columns of the spreadsheet are as follows:

District (Column 1): This column identifies the district number of the county in column 3.

County_id (Column 2): Column 1 and column 2 together form an unique identifier for the counties in column 3. A county is identified by its County_id (column 2) and the District (column 1) it belongs to. This was specifically needed to identify and differentiate the counties that belonged to more than one district. The software considers the counties belonging to different districts as separate entities.

County (Column 3): This column contains the name of the county.

Republicans and Democrats (Column 4): This contains the total number of votes in favor of the Republican party (GOP) and the Democratic party in the county identified by Column 1 and Column 2.

Neighbors (Column 5): This column contains information about the "neighboring counties" of the given county. Neighboring counties represent the counties that share a boundary with the county identified by Column 1 and Column 2. Individual neighbors are separated by commas.

7.3 The local-search heuristic

Informally, our algorithm starts with the existing (possibly gerrymandered) districts and then repeatedly attempts to *reassign* counties (or parts of counties) into neighboring districts. This was done on a *semi-random* basis, and on average about 100 iterations were carried out in each run. Each time a county (or a part of a county) was shifted, the efficiency gap was calculated to check if it was less than the prior efficiency gap. Details of our algorithm are shown in Fig. 7.

We cannot provide *any* theoretical analysis of the randomized algorithm in Fig. 7 because *no* such analysis is possible (due to Theorem 1) as stated formally in the following lemma.

Lemma 5 *Assuming* $\mathsf{P} \neq \mathsf{NP}$ *(respectively,* $\mathsf{RP} \neq \mathsf{NP}$*), there exists no deterministic local-search algorithms (respectively, randomized local-search algorithms) that reaches a solution with a finite approximation ratio in polynomial time starting at any non-optimal valid solution.*

[12] As commonly done by researchers in gerrymandering of two-party systems, we ignore negligible "third-party" votes, *i.e.*, votes for candidates other than the democratic and republican parties.

```
start with the current districts, say 𝒬_1, ..., 𝒬_κ
repeat μ times (* μ was set to 100 in actual run *)
    select a random r ∈ {0,1,...,k} for some 0 < k < |V|   (* k = 20 in actual run *)
    select r nodes v_1, ..., v_r from 𝔊 at random
                        (* Note that a node is a county or part of a county *)
    counties_done ← ∅
    for each v_i do
        if all neighbors of v_i do not belong to the same district as v_i then
            if v_i ∉ counties_done then
                add v_i to counties_done
                for every neighbor v_j of v_i do
                    if assigning v_i to the district of v_j produces no district with
                      disconnected parts then
                        assign v_i to the district of one of its neighbors
                        recalculate new districts, say 𝒬'_1, ..., 𝒬'_κ
                        if min_{1≤i≤κ} {Pop(𝒬_i)} ≤ Pop(𝒬'_j) ≤ max_{1≤i≤κ} {Pop(𝒬_i)} for every j
                         then
                            if Effgap_κ(𝒫, 𝒬'_1, ..., 𝒬'_κ) < Effgap_κ(𝒫, 𝒬_1, ..., 𝒬_κ)
                             then
                                    𝒬_1 ← 𝒬'_1 ; 𝒬_2 ← 𝒬'_2 ; ...... ; 𝒬_κ ← 𝒬'_κ
                            end_if
                        end_if
                    end_if
                end_for
            end_if
        end_if
    end_for
end_repeat
```

Fig. 7: A local search algorithm for computing efficiency gap; comments are enclosed within ($*$ and $*$). The algorithm was implemented using the PYTHON language.

Proof. This follows from the proof of Theorem 1 once observes that the specific instance of the MIN-WVP$_\kappa$ problem created in the reduction of the theorem has exactly one (trivial) non-optimal solution and every other valid solution is an optimal solution. ❑

	Vote share		Number of Seats		Normalized efficiency gap
	Democrats	**GOP**	**Democrats**	**GOP**	(current)
	$\frac{\mathsf{PartyA}(\mathscr{P})}{\mathsf{Pop}(\mathscr{P})}$	$\frac{\mathsf{PartyB}(\mathscr{P})}{\mathsf{Pop}(\mathscr{P})}$			$\mathsf{Effgap}_\kappa(\mathscr{P},\dots\dots)/\mathsf{Pop}(\mathscr{P})$
Wisconsin	50.75%	49.25%	3	5	14.76%
Texas	43.65%	56.35%	12	24	4.09%
Virginia	51.96%	48.04%	4	7	22.25%
Pennsylvania	50.65%	49.35%	5	13	23.80%

Table 1: Summary statistics for 2012 election data for election of the (federal) house of representatives for the states of Texas, Wisconsin, Virginia and Pennsylvania.

7.4 Empirical evaluations of algorithm in Fig. 7 and corresponding implications

Our resulting software based on the algorithm in Fig. 7 was tested on four real electoral data for the 2012 election of the (federal) house of representatives for the US states of Wisconsin [42, 43], Texas [44, 45], Virginia [40, 41] and Pennsylvania [38, 39]. Some summary statistics for these data are shown in Table 1. The results of running the local-search algorithm in Fig. 7 on the four real data-sets are tabulated in Table 2, and the corresponding maps are shown in Fig. 8–11. *The results computed by our algorithm are truly outstanding: the final efficiency gap was lowered to* 3.80%, 3.33%, 3.61% *and* 8.64% *from* 14.76%, 4.09%, 22.25% *and* 23.80% *for Wisconsin, Texas, Virginia and Pennsylvania, respectively, in a small amount of time*. Our empirical results clearly show that it is very much possible to design and implement a very fast algorithm that can "un-gerrymander" (*based on the efficiency gap measure*) the gerrymandered US house districts of four US states.

	Number of Seats				Normalized efficiency gap	
	Original		New		$\mathsf{Effgap}_\kappa(\mathscr{P},\dots\dots)/\mathsf{Pop}(\mathscr{P})$	
	Democrats	**GOP**	**Democrats**	**GOP**	**Original**	**New**
Wisconsin	3	5	3	5	14.76%	3.80%
Texas	12	24	12	24	4.09%	3.33%
Virginia	3	8	5	6	22.25%	3.61%
Pennsylvania	5	13	6	12	23.80%	8.64%

Table 2: Redistricting results obtained by running the algorithm in Fig. 7 for the states of Texas, Wisconsin, Virginia and Pennsylvania in comparison to the 2012 district plans.

A closer look at the new district maps shown in Fig. 8–11 also reveal the following interesting insights:

Seat gain vs. efficiency gap. Lowering the efficiency gap from 15% to 3.75% for the state of Wisconsin did *not* affect the total seat allocation (3 democrats vs. 5

republicans) between the two parties. Indeed, this further reinforces the assertion in [36] that *the efficiency gap and partisan symmetry are different concepts*, and thus fewer absolute difference of wasted votes does not necessarily lead to seat gains for the loosing party.

Compactness vs. efficiency gap. The new district maps for the state of Virginia reveals an interesting aspect. Our new district map have fewer districts that are oddly shaped compared to the map used for the 2012 election[13], even though minimizing wasted votes does not take into consideration shapes of districts.

How natural are gerrymandered districts? Since our algorithm applies a sequence of carefully chosen semi-random *perturbations* to the original gerrymandered districts to drastically lower the absolute difference of wasted votes, one can *hypothesize* that the original gerrymandered districts are far from being a product of *arbitrarily random* decisions. However, to reach a definitive conclusion regarding this point, one would need to construct a suitable null model, which we do not have yet.

8 Conclusion and future research

In this article we have performed algorithmic analysis of the recently introduced efficiency gap measure for gerrymandering both from a theoretical (computational complexity) as well as a practical (software development and testing on real data) point of view. The main objective of the paper was to provide a scientific analysis of the efficiency gap measure and to provide a crucial supporting hand to remove partisan gerrymandering should the US courts decide to recognize efficiency gap as at least a partially valid measure of gerrymandering. Of course, final words on resolving gerrymandering is up to the US judicial systems. The following research questions may be of interest to future investigators of the science of gerrymandering:

- ▷ Formulate and investigate the computational complexity properties of a measure of gerrymandering that *combines* the efficiency gap with other aspects such as compactness in the objective function.
- ▷ What is the precise computational complexity (*i.e.*, NP-completeness, existence of non-trivial polynomial-time approximation algorithms, *etc.*) of the MIN-WVP$_\kappa$ problem when the total population $\sum_{i,j} \text{Pop}_{i,j}$ is polynomial in $|\mathscr{P}|$? We *conjecture* that MIN-WVP$_\kappa$ is NP-complete even for this special case, but have been unable to prove so.

We hope that our research work and software will provide a crucial *supporting hand* to remove partisan gerrymandering. However, the goal of writing article should not be viewed to have the final word on gerrymandering, but to introduce a series of concepts, models and problems and to show that science of gerrymandering involves an intriguing set of partitioning problems involving geometric and combinatorial optimization.

[13] Virginia is one of the most gerrymandered states in the country, both on the congressional and state levels, based on lack of compactness and contiguity of its districts. Virginia congressional districts are ranked the 5$^{\text{th}}$ worst in the country because counties and cities are broken into multiple pieces to create heavily partisan districts [48].

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Conflict of interest

The authors declare that they have no conflict of interest.

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(A)

(B)

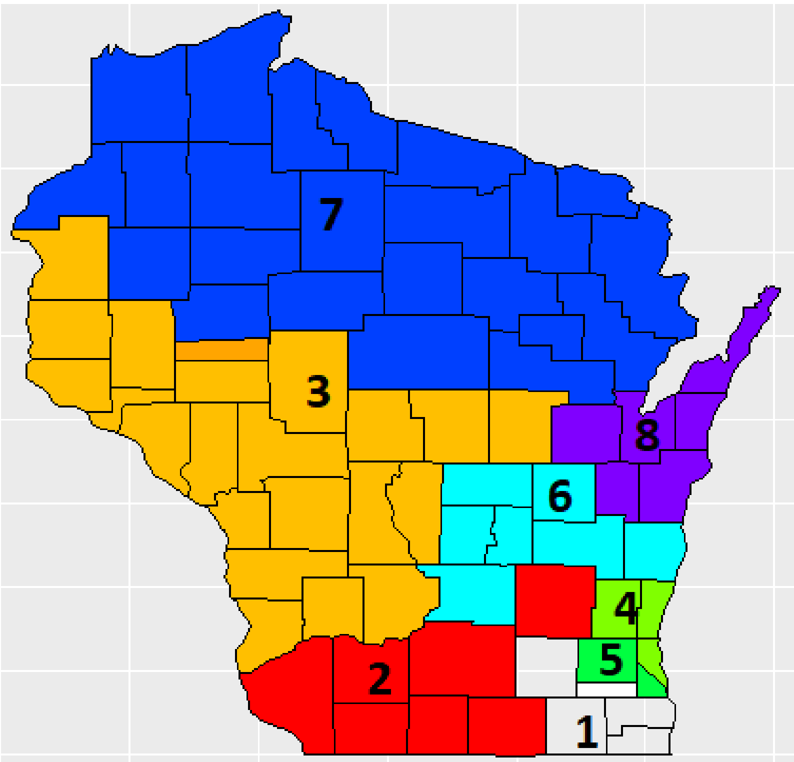
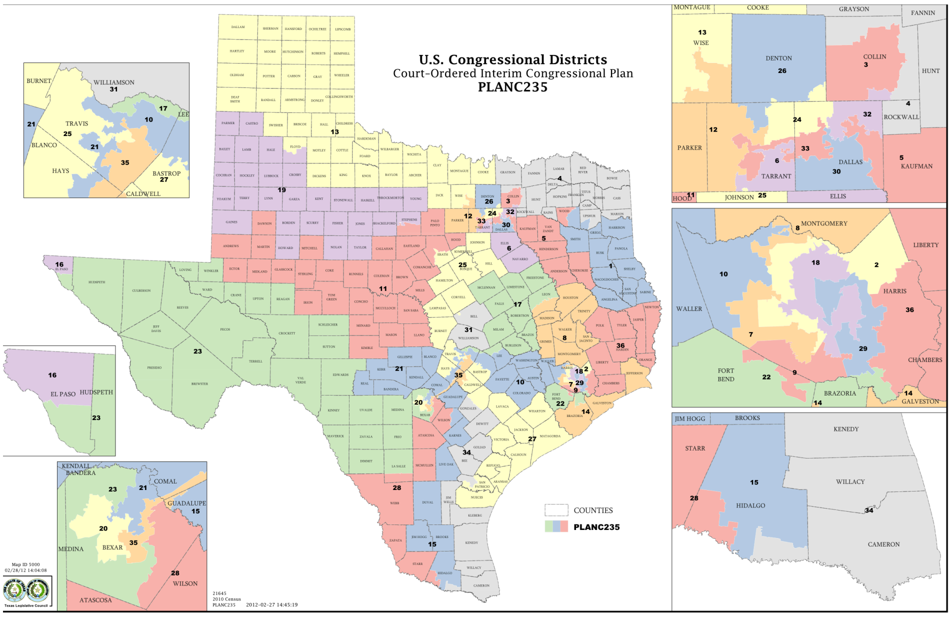
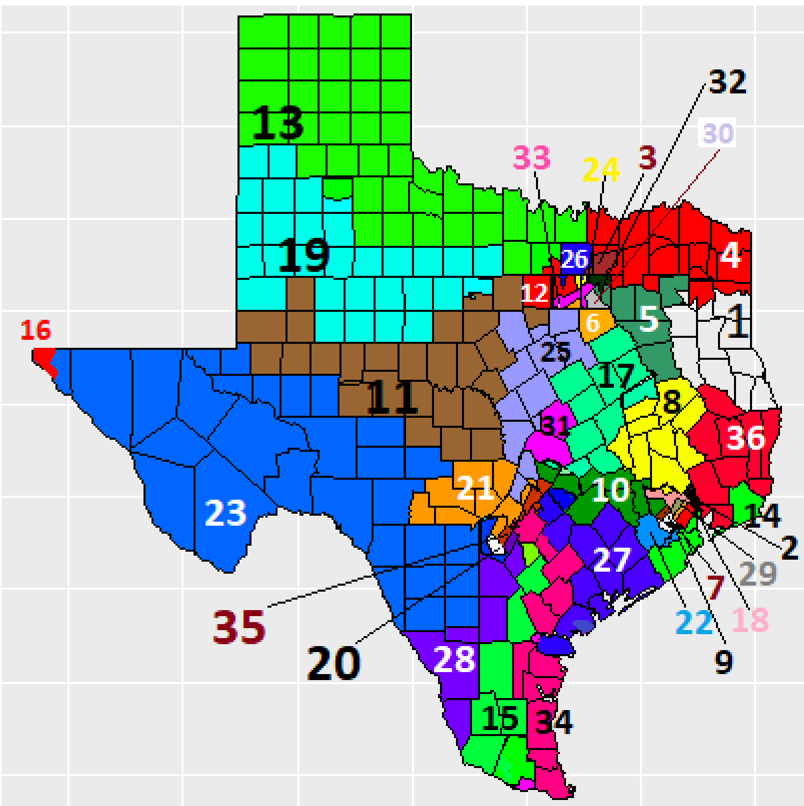


Fig. 8: The district maps of Wisconsin: **(A)** original [43] and **(B)** after applying our local search algorithm in Fig. 7. The efficiency gap was reduced from 14.76% in **(A)** to 3.80% in **(B)**.

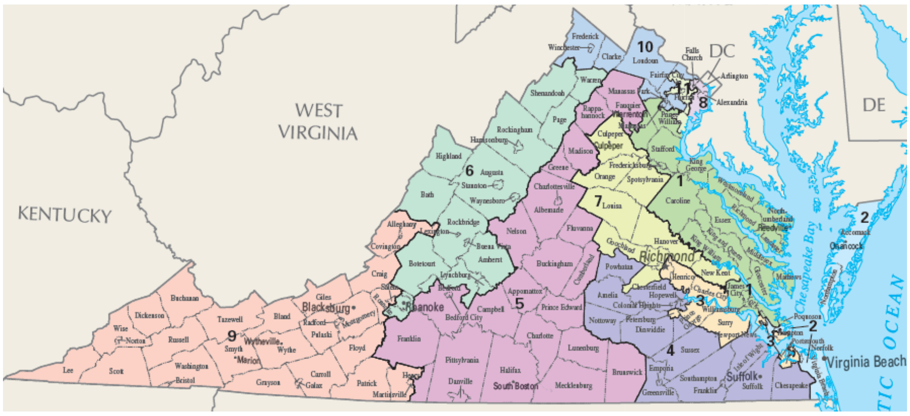


(A)

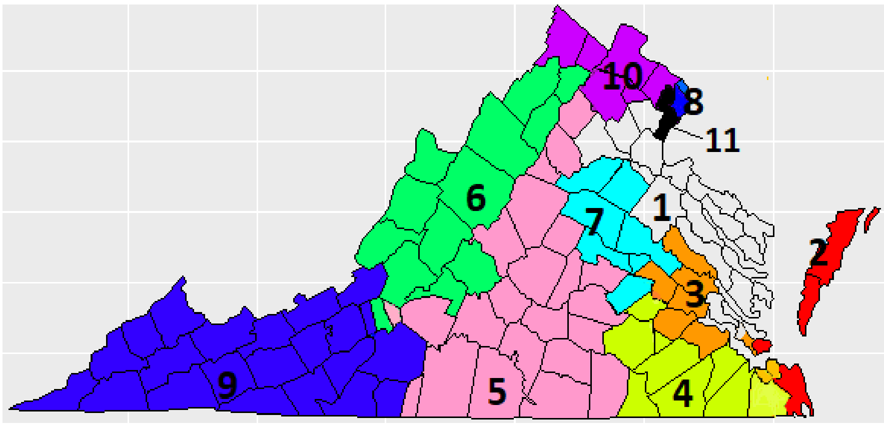


(B)

Fig. 9: The district maps of Texas: **(A)** original [45] and **(B)** after applying our local search algorithm in Fig. 7. The efficiency gap was reduced from 4.09% in **(A)** to 3.33% in **(B)**.

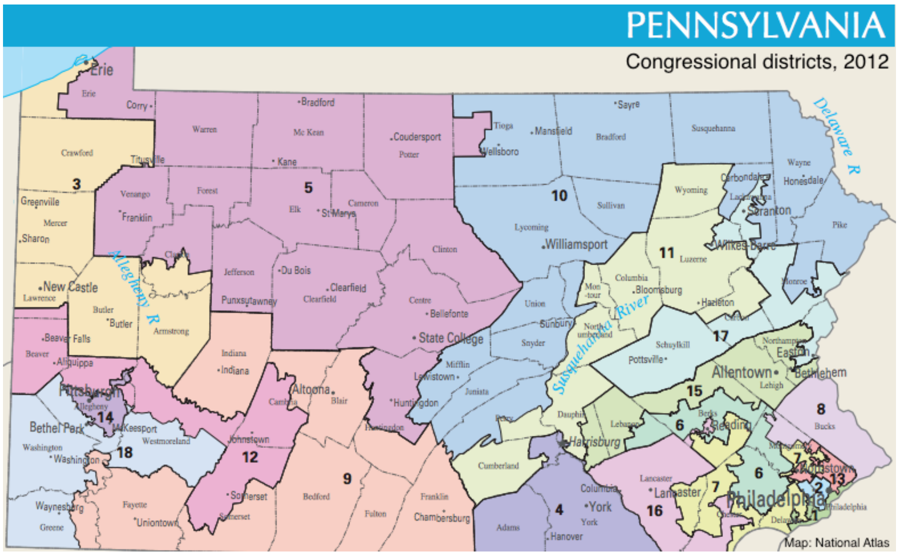


(A)

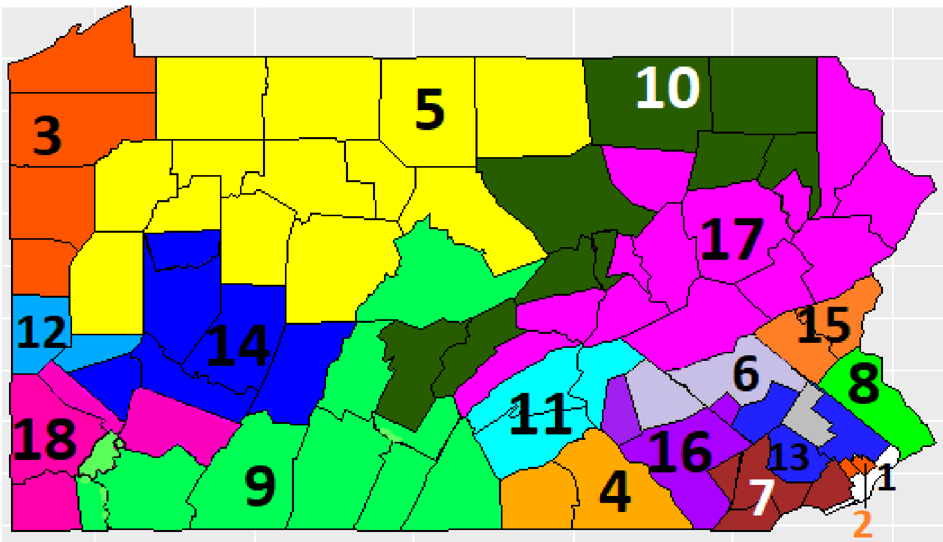


(B)

Fig. 10: The district maps of Virginia: **(A)** original [41] and **(B)** after applying our local search algorithm in Fig. 7. The efficiency gap was reduced from 22.25% in **(A)** to 3.61% in **(B)**.



(A)



(B)

Fig. 11: The district maps of Pennsylvania: **(A)** original [39] and **(B)** after applying our local search algorithm in Fig. 7. The efficiency gap was reduced from 23.80% in **(A)** to 8.64% in **(B)**.