Characterizations of Bipartite Steinhaus Graphs

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Abstract

We characterize bipartite Steinhaus graphs in three ways by partitioning them into four classes and we describe the color sets for each of these classes. An interesting recursion had previously been given for the number of bipartite Steinhaus graphs and we give two fascinating closed forms for this recursion. Also, we exhibit a lower bound, which is achieved infinitely often, for the number of bipartite Steinhaus graphs.

Key words: Steinhaus graph, bipartite Steinhaus graph, recursive sequence

1 Introduction

Let $T = a_{0,0}a_{0,1}\ldots a_{0,n-1}$ be an n-long string of 0s and 1s beginning with 0. The *Steinhaus graph* generated by T has as its adjacency matrix the *Steinhaus matrix* $A = [a_{i,j}]$, where

$$a_{i,j} = \begin{cases} 0, & \text{if } 0 \le i = j \le n-1; \\ (a_{i-1,j-1} + a_{i-1,j}) \bmod 2, & \text{if } 0 < i < j \le n-1; \\ a_{j,i}, & \text{if } 0 \le j < i \le n-1. \end{cases} \tag{1}$$

As illustrated in Figure 1, a vertex of a Steinhaus graph is usually labeled by its corresponding row number. A *Steinhaus triangle* is the upper-triangular part of a Steinhaus matrix (excluding the diagonal); hence it is generated by a string of length $n-1$. Throughout this paper, n will always be the size of the vertex sets of the graphs under discussion and thus there are exactly 2^{n-1} Steinhaus graphs of size n.

$$\begin{array}{c c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

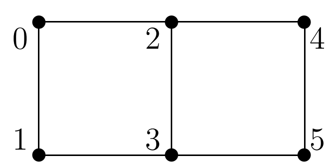

Fig. 1. Example of a Steinhaus matrix and graph

We say that $a_{0,0}a_{0,1}\ldots a_{0,n-1}$ is a *(row) generator* of the corresponding Steinhaus graph. This definition violates a fundamental principle in doing mathematics. If a problem has any kind of symmetry, the mathematical structures defined to handle that problem should reflect that symmetry. In our case, there is just one symmetry. The vertices are ordered $0, \ldots, n-1$, but the reverse order $n-1, \ldots, 0$ is equally good. There is no reason to distinguish the first vertex from the last vertex of the graph by starting with the first

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row rather than the last column of the adjacency matrix. Selecting the string $a_{0,1}a_{1,2}\ldots a_{n-2,n-1}$ as the "name" of a Steinhaus graph and defining the matrix in (2) gives a symmetric way of creating a Steinhaus graph. We say that the string $a_{0,1}a_{1,2}\ldots a_{n-2,n-1}$ *diagonally generates* its graph and so the graph in Figure 1 is diagonally generated by 10101. A diagonal generator reflects the structural properties of its graph. In particular, bipartite Steinhaus graphs are characterized by some simple patterns in their diagonal generators.

$$a_{i,j} = \begin{cases} 0, & \text{if } 0 \leq i = j \leq n-1; \\ (a_{i,j-1} + a_{i+1,j}) \bmod 2, & \text{if } 0 \leq i < j-1 < n-1; \\ a_{j,i}, & \text{if } 0 \leq j < i \leq n-1. \end{cases} \tag{2}$$

Another way to describe a Steinhaus graph is to let minAdj(0) be the vertex with the smallest label that is adjacent to vertex 0 and $\mathrm{Adj}^+(v)$ be the set of all vertices with labels larger than the vertex labeled v that are adjacent to v. In Figure 1, $\mathrm{minAdj}(0) = 1$ and $\mathrm{Adj}^+(1) = \{3\}$. It is easy to show

Proposition 1 *A Steinhaus graph is determined uniquely if* $v = \mathrm{minAdj}(0)$ *and the set* $\mathrm{Adj}^+(v)$ *are given.*

Steinhaus graphs and triangles are named after Hugo Steinhaus who asked in [18] if there are Steinhaus triangles containing the same number of 0s and 1s. Harborth [13] answered this in the affirmative by showing that for each n congruent to 0 or 1 mod 4, there are at least four strings of length $n-1$ that generate such triangles. Wang [19] named these triangles after Steinhaus (see also [14]) and Chang [6] investigated the possible number of 1s in these triangles. Molluzzo [17] was the first to form graphs from Steinhaus triangles, but he examined the complements of what we call Steinhaus graphs. For a survey of Steinhaus graphs see [9] which also announces some of the results in this paper.

Steinhaus graphs form a "large" enough class of graphs to be interesting. Brigham and Dutton [5] conjectured that almost all Steinhaus graphs have diameter two. Brand [1] verified this for both Steinhaus graphs and their complements and his results were generalized in [2]. Brigham et al. [4] have shown the following result.

Fact 1 *Every graph is an induced subgraph of a Steinhaus graph.*

Steinhaus graphs are also "large" in the sense that they mimic the behavior of all graphs (at least in terms of first order properties).

Fact 2 *For a given first order property, almost all Steinhaus graphs have the property if and only if almost all graphs have the property.*

Using this fact, the authors of [3] show that almost all Steinhaus graphs have the following properties: They are k-connected for any fixed k, they contain any fixed graph as a subgraph, and they are not planar.

2 Notation and bipartite Steinhaus graphs

In this paper we use the following notation and definitions.

(1) For $b < a$, let $\binom{b}{a} = 0$.
(2) Let $P_{i,j}$ be $\left\{\binom{i}{k} \bmod 2\right\}_{k=0}^{j-1}$ which is a j-long string of 0s and 1s. Note that $P_{i,i+1}$ is the ith row (0-origin) of Pascal's triangle modulo 2. We use the name Pascal's rectangle (see Figure 2) for a zero-padded version of Pascal's triangle modulo 2.
(3) If T is a string of 0s and 1s, then T^s is the string repeated s times. For example, $0(P_{1,3})^3 = 0110110110$. Also, T^∞ is an infinite repetition of T.
(4) Denote the set of integers by $\mathbb{Z}$ and if $\mathbb{A} \subseteq \mathbb{Z}$, then $\mathbb{A}_i = \{x \in \mathbb{A}\colon\ x \geq i\}$. For example, $\mathbb{N} = \mathbb{Z}_0$ is the set of non-negative integers, $\mathbb{Z}^+ = \mathbb{Z}_1$ is the set of positive integers, and if $\mathbb{O}$ is the set of odd positive integers, then $\mathbb{O}_3$ is the set of odd positive integers larger than 1.
(5) As is usual, $\lfloor x \rfloor$ is the floor of x and $\lceil x \rceil$ is the ceiling of x.
(6) We denote $\log_2(x)$ by $\lg(x)$.
(7) If a variable is a positive integer, we will use the capital of that variable to be the next larger power of two. If $k \in \mathbb{Z}^+$, then $K = 2^{\lceil \lg(k) \rceil}$. For example, if $k = 5$, then $K = 8$ and if $v = 4$, then $V = 4$.
(8) Let $n_1 = n - 1$ and $n_2 = n - 2$.
(9) If u is odd and $v \in \mathbb{N}$, then

$$f(u2^v) = \begin{cases} 2^v, & \text{if } u \in \mathbb{O}_3; \\ 2^v + \frac{1}{2}, & \text{if } u = 1. \end{cases} \tag{3}$$

$$\begin{array}{cccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \dots \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & \dots \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \dots \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Fig. 2. Pascal's rectangle

There are two facts concerning Pascal's rectangle that we need and which are corollaries of Lucas's theorem. See [10] for a proof of these facts and [11] for a short proof of Lucas's theorem.

Fact 3 (Lucas's Theorem) *Let p be prime and $m = m_0 + m_1 p + m_2 p^2 + \cdots + m_r p^r$ and $k = k_0 + k_1 p + k_2 p^2 + \cdots + k_r p^r$ with $0 \le m_i < p$ and $0 \le k_i < p$ for $0 \le i \le r$. Then*

$$\binom{m}{k} \equiv \binom{m_0}{k_0}\binom{m_1}{k_1}\cdots\binom{m_r}{k_r} \pmod{p}.$$

Fact 4 *For $m \in \mathbb{N}$ and $k \le 2^m$, $P_{2^m-1,k} = 1^k$ and $P_{2^m,2^m+1} = 1(0^{2^m-1})1$.*

Fact 5 *If $2^{m-1} < k \le 2^m$, then $P_{2^m+j,k} = P_{j,k}$ and so, for k fixed, $P_{j,k}$ is periodic of period 2^m.*

In [8], it was shown that a bipartite Steinhaus graph has a perfect matching if and only if the sizes of the two color sets are equal and that a Steinhaus graph is bipartite if and only if the graph contains no triangles. This is another example of how Steinhaus graphs mimic the behavior of all graphs since Györi et al. [12] showed that graphs without short odd cycles are nearly bipartite. [They proved that for every constant $\epsilon > 0$ and every graph G with n vertices that contains no odd cycles of length smaller than ϵn, G can be made bipartite by removing $(15/\epsilon)\log(10/\epsilon)$ vertices (this is the natural log).]

In a follow-up to [8], Dymàček and Whaley [10] characterized the generating strings of bipartite graphs (called *bipartite strings*) by showing that all Steinhaus graphs are generated by the strings described in Facts 6 and 7 below. If T is a bipartite string but $T0$ and $T1$ are both non-bipartite strings, then T is called a *maximal* bipartite string. They also gave the following recurrence for the number $b(n)$ of n-long bipartite strings: $b(2) = 2$, $b(3) = 4$, and for $k \ge 2$,

$$b(2k+1) = 2b(k+1) + 1 \quad \text{and} \quad b(2k) = b(k) + b(k+1). \tag{4}$$

Furthermore, they proved that $b(n) \le (5n-7)/2$ with equality occurring when n is one more than a power of 2.

Fact 6 (Periodic characterization) *For $s \in \mathbb{Z}^+$ and for $t, v \in \mathbb{N}$, any string of the form $0^s(P_{S-s,S})^{2^v}0^t$ is a bipartite string.*

Fact 7 (Maximal length characterization) *For $s \in \mathbb{Z}^+$, $u \in \mathbb{O}_3$, and $v \in \mathbb{N}$, a string of the form $0^s(P_{S-s,S})^{u2^v}0^{S2^v}$ is a maximal bipartite string.*

3 Diagonal generators, adjacency list, and color sets

In this section we give a simple characterization of the diagonal generators and adjacency lists, an alternative description of the row generators, and lists of the possible color sets of bipartite Steinhaus graphs. To do this, we separate the bipartite Steinhaus graphs into four classes which are listed in Definition 2. Classes 1 and 2 are associated with the infinite bipartite graphs that are diagonally generated by sequences of the form $(P_{0,2^r})^\infty$ where $r \in \mathbb{N}$. Class 3 is derived from the diagonal strings which cannot be infinitely extended. Since all Steinhaus graphs are connected except for the null graph, see [7], the color sets for a given bipartite graph are unique. In the following definition, we use our convention that $S = 2^{\lceil \lg(s) \rceil}$ and the row generators are from Facts 6 and 7. Note that basically all bipartite strings have the form $0^s(P_{S-s,s})^{u2^v}0^t$.

Definition 2 (The four classes) *Let $s, t \in \mathbb{Z}^+$, $v \in \mathbb{N}$, and $u \in \mathbb{O}_3$.*

Class 0 (The null graph) *The row generator is 0^n.*
Class 1 ($\mathbf{t = 0}$) *A row generator for a graph in Class 1 is an n-long prefix of $0^s(P_{S-s,S})^{2^v}$.*
Class 2 ($\mathbf{t > 0}$) *A row generator for a graph in Class 2 is an n-long prefix of $0^s(P_{S-s,S})^{2^v}0^t$, where we require that $s + S2^v < n$.*
Class 3 (The maximal bipartite strings) *A row generator for a graph in Class 3 is an n-long prefix of $0^s(P_{S-s,S})^{u2^v}0^{S2^v}$, where we require that $s + uS2^v < n$.*

In Theorems 3 to 6, we give for each class other than Class 0 another description of the row generator, a characterization of the diagonal generators and adjacency lists, and the color sets for the graphs.

Theorem 3 (Class 0) *The graph in Class 0 is also described by the following:*

- *Row generator:* 0^n.
- *Diagonal generator:* 0^{n-1}.
- *Adjacency list:* $\mathrm{minAdj}(0)$ *is undefined and* $\mathrm{Adj}^+(0) = \varnothing$.
- *Color sets: Any will do.*

For Theorems 4 and 5, please refer to Figure 3 where the matrix before the double vertical line represents a typical matrix in Class 1 and the entire matrix represents a typical matrix in Class 2. Note that the diagonal elements are underlined. Whenever we describe a row of a Steinhaus matrix, we list only the diagonal element and the elements to the right of the diagonal.

Theorem 4 (Class 1) *Let $s \in \mathbb{Z}^+$ and $v \in \mathbb{N}$. The graphs in Class 1 are also described by the following:*

$$
\begin{array}{ccc|cccc|cccc||cccc|c}
0 & & & s & & & & & & & n_1 & & & & & n_1 \\
\underline{0} & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & \underline{0} & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & \underline{0} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\cline{4-16}
 & & & \underline{0} & & & & & & & & 1 & 0 & 0 & 0 & 0 \\
 & & & & \underline{0} & & & & & & & 1 & 1 & 0 & 0 & 0 \\
 & & & & & \underline{0} & & & 0 & & & 1 & 0 & 1 & 0 & 0 \\
 & & & & & & \underline{0} & & & & & 1 & 1 & 1 & 1 & 0 \\
 & & & & & & & \underline{0} & & & & 1 & 0 & 0 & 0 & 1 \\
 & & & & & & & & \underline{0} & & & 1 & 1 & 0 & 0 & 1 \\
 & & & & & & & & & \underline{0} & & 1 & 0 & 1 & 0 & 1 \\
 & & & & & & & & & & \underline{0} & 1 & 1 & 1 & 1 & 1 \\
\cline{12-16}
 & & & & & & & & & & & \underline{0} & 0 & 0 & 0 & 0 \\
 & & & & & & & & & & & & \underline{0} & 0 & 0 & 0 \\
 & & & & & & & & & & & & & \underline{0} & 0 & 0 \\
 & & & & & & & & & & & & & & \underline{0} & 0 \\
 & & & & & & & & & & & & & & & \underline{0}
\end{array}
$$

Fig. 3. $s = 3$: Class 1, $n_1 = 10$; Class 2, $n_1 = 15$

- *Alternative description of row generator:* $0^s P_{N-s,n-s}$.
- *Diagonal generator:* $0^{s-1}10^{n-s-1}$.
- *Adjacency lists:* $\text{minAdj}(0) = s$, $\text{Adj}^+(s) = \varnothing$.
- *Color sets:* $\big\{\{0, 1, \ldots, s-1\}, \{s, s+1, \ldots, n-1\}\big\}$.

PROOF. If $0^s(P_{S-s,S})^{2^v}$ is a row generator, then by Fact 4, row $s-1$ is $\underline{0}(P_{S-1,S})^{2^v} = \underline{0}1^{S2^v}$. If $0^s P_{N-s,n-s}$ is an alternative row generator, then again by Fact 4, row $s-1$ is $\underline{0}P_{N-1,n-s} = \underline{0}1^{n-s}$. For each of these generators, row $s-1$ is all 1s (from the diagonal right) and so row s is all 0s. Hence $\text{minAdj}(s) = 0$ and $\text{Adj}^+(s) = \varnothing$ for both generators. Thus these generators produce the same matrix and are therefore the same string. Also, note that if $\text{minAdj}(s) = 0$ and $\text{Adj}^+(s) = \varnothing$, then the matrix is generated by $0^s P_{N-s,n-s}$.

Now these generators give exactly one 1 on the diagonal generator because $a_{i,s} = 1$ for $0 \le i < s$ and $a_{s-1,j} = 1$ for $s \le j < n$. Conversely, it is easy to see that if there is just one 1 in the diagonal generator at position $a_{s-1,s}$, then row $s-1$ is $\underline{0}1^{n-s}$ and hence row 0 is $0^s P_{N-s,n-s}$.

Since $\text{Adj}^+(s) = \varnothing$ and vertex s is adjacent to each of the vertices in the set $\{0, 1, \ldots, s-1\}$, the color classes are as given in the statement of the theorem. $\square$

Theorem 5 is illustrated in Figure 3 with $n = 16$, $v = 1$, and $r = s = 3$. Note that if the matrix were larger, the period of the diagonal generator would be 8.

Theorem 5 (Class 2) *Let $s, t \in \mathbb{Z}^+$ and $r, v \in \mathbb{N}$. If $1 \le s \le 2^r$ and $s + 2^r <$*

n, then the graphs in Class 2 are also described by the following:

- *Alternative description of row generator:* $0^s P_{2^r-s,n-s}$.
- *Diagonal generator: A sequence with two or more* 1*s separated by exactly* (2^r-1) 0*s; i.e., an $(n-1)$-long prefix of* $0^{s-1}(P_{0,2^r})^\infty$ *containing more than one* 1.
- *Adjacency lists:* $\mathrm{minAdj}(0) = s$, $\mathrm{Adj}^+(s) = \{s+2^r\}$.
- *Color sets: Let v be a vertex, so $0 \le v < n$. Compute q_v by $v-s = q_v 2^r + w$ with $0 \le w < 2^r$. Then the color sets are $\{v : q_v$ is even$\}$ and $\{v : q_v$ is odd$\}$.*

PROOF. Consider row $s-1$ of the matrices generated by both the known and the alternative generators. For the known generator, row $s-1$ is $\underline{0}1^{S2^v}0^t = \underline{0}1^{S2^v}0^{n-s-S2^v}$ and for the alternative generator row $s-1$ is $\underline{0}1^{2^r}0^{n-s-2^r}$.

Let s, t, and v be as given in the definition of Class 2 and let $r = \lceil \lg(s) \rceil + v$. So $s \le S \le S2^v = 2^r$. Since $\underline{0}1^{S2^v}0^{n-s-S2^v} = \underline{0}1^{2^r}0^{n-s-2^r}$, $\mathrm{Adj}^+(s) = \{s+S2^v\} = \{s+2^r\}$ for both generators and hence the matrices are the same.

Given r and s with $1 \le s \le 2^r$ and $s + 2^r < n$, consider the alternative generator $0^s P_{2^r-s,n-s}$. Note that the condition $s+2^r < n$ guarantees that $\mathrm{Adj}^+(s)$ is not empty, which distinguishes Class 2 generators from Class 1 generators. Let $v = r - \lceil \lg(s) \rceil$. So $2^r = S2^v$. For the matrix generated by the alternative generator, row $s-1$ is $\underline{0}1^{2^r}0^{n-s-2^r}$ which is the same as row $s-1$ in the matrix generated by $0^s(P_{S-s,S})^{2^v}0^{n-1-S2^v}$. Hence, this generator generates the same matrix as the alternative generator.

By the previous paragraph, deleting the first s rows of the matrix generated by $0^s P_{2^r-s,n-s}$ gives a matrix with first row $0^{2^r}10^{n-s-2^r}$. Hence, the part of the matrix bounded by columns $s+2^r$ to $s+2^{r+1}-1$ and rows s to $s+2^r-1$ is the first 2^r rows and columns of Pascal's rectangle. As is easily seen, this propagates endlessly giving a diagonal generator that is the first $n-1$ entries of $0^{s-1}(P_{0,2^r})^\infty$.

Conversely, using the first $n-s$ terms of $(P_{0,2^r})^\infty$ as a diagonal generator gives the matrix whose first row is $01^{2^r}0^{n-s-2^r}$. Adding s rows above this row with $\mathrm{minAdj}(0) = s$ gives the generator $0^s P_{2^r-s,n-s}$ and therefore, using the first $n-1$ terms of $0^{s-1}(P_{0,2^r})^\infty$ as a diagonal generator gives the row generator $0^s P_{2^r-s,n-s}$.

The color sets are determined easiest by considering the periodic nature of the diagonal generators. Clearly, $\{0,\ldots,s-1\}$ must be in one set since each is adjacent to the vertex labeled s. The periodic nature of the generator gives alternating sets of 2^r vertices in each color set. $\square$

The remaining class of bipartite Steinhaus graphs is generated by prefixes of maximal length bipartite strings. These are maximal because if $n = s + uS2^v + S2^v$, then $T = 0^s(P_{S-s,S})^{u2^v}0^{S2^v}$ is a bipartite string but neither $T0$ nor $T1$ are bipartite. In Figure 4, $s = 2$, $u = 3$, and $v = 1$ and note that if $n = 19$, then vertices 6, 14, and 18 form a triangle. In general, the vertices labeled $s + S2^v$, $s + uS2^v$, and n form a triangle in $T0$ and the vertices labeled 0, s, and n form a triangle in $T1$. The strings in Class 3 are not difficult to describe, but are difficult to count. The number of such strings satisfies a recursion that has two closed forms which are somewhat bizarre. Theorem 6 is illustrated in Figure 4 where $w = 2$ and $x = s + u2^w = 14$.

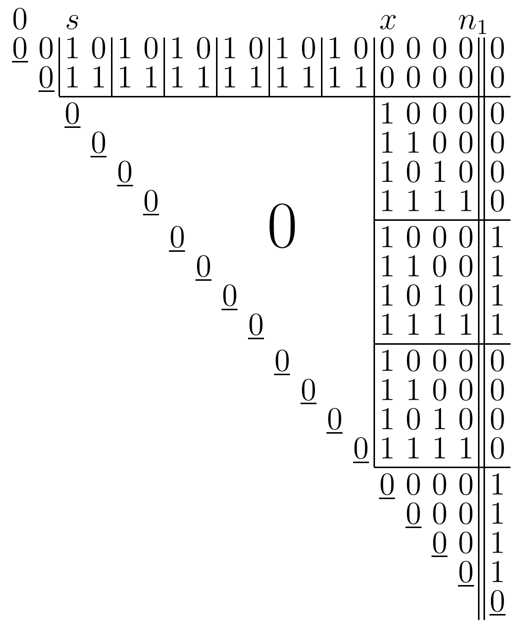

Fig. 4. Class 3: $n_1 = 17$, $s = 2$, $u = 3$, $v = 1$

Theorem 6 (Class 3) *Let $s \in \mathbb{Z}^+$, $v, w \in \mathbb{N}$, and $u \in \mathbb{O}_3$. If s, u, and w are such that $1 \le s \le 2^w$, $x = s + u2^w$, and $0 < n - x \le 2^w$, then the graphs in Class 3 are also described by the following:*

- *Alternative description of row generator: An n-long prefix of $0^s(P_{2^w-s,2^w})^u0^{n-x}$.*
- *Diagonal generator: An $(n-1)$-long prefix of $0^{s-1}10^{u2^w-1}10^{2^w-1}$ containing two 1s.*
- *Adjacency lists:* $\mathrm{minAdj}(0) = s$, $\mathrm{Adj}^+(s) = \{x\}$.
- *Color sets:* $\big\{\{0, 1 \ldots, s-1\} \cup \{x, x+1, \ldots, n-1\}, \{s, s+1, \ldots, x-1\}\big\}$.

PROOF. If $s = 1$, then $P_{S-s,S} = 1$ and so the row generator $01^{n-2}0$ is in Class 3, unless $n = 2^v + 2$ for $v \in \mathbb{Z}^+$ in which case it is in Class 2.

As in the proof of Theorem 5, the known and the alternative generators each give $\text{Adj}^+(s) = \{x\}$ since we let $2^w = S2^v$. The restriction on $n - x$ simply forces the alternative generator to have at least one trailing 0, but not enough trailing 0s to cease being a bipartite string. Therefore, each of these generators generates the same matrix. Neither is in Class 1 or Class 2 because $u \in \mathbb{O}_3$.

For the matrix generated by the diagonal generator, consider the triangle of 0s formed by the 0s between the two 1s in the diagonal generator. Note that the top border of this triangle consists of $u2^v$ S-size blocks of Pascal's rectangle and the right border consists of u 2^w-size blocks of Pascal's rectangle (see Figure 4). Hence for the diagonal generator, $\text{Adj}^+(s) = \{x\}$.

For either the alternative row generator or the diagonal generator, row $s - 1$ is $\underline{0}1^{u2^w}10^{n-x}$ and so there are no edges in the graph induced by the vertices $\{s, s+1, \ldots, x-1\}$. Columns x to $n-1$, starting with row s, is Pascal's rectangle. By Fact 5 and the hypothesis $n - x \leq 2^w$, row x is all 0s to the right of the diagonal. Because of this and $\text{minAdj}(0) = s$, the graph induced by the vertices $\{0, 1, \ldots, s-1, x, x+1, \ldots, n-1\}$ has no edges. $\square$

Theorems 7, 8, and 10 restate some of the results from Theorems 3 to 6. Theorem 9 is a corollary of the diagonal characterizations found in the same theorems. A different proof of Theorem 10 can be found in [15] and [16].

Theorem 7 (Diagonal generators) *Let $u \in \mathbb{O}_3$, $s \in \mathbb{Z}^+$, and $v \in \mathbb{N}$. A Steinhaus graph is bipartite if and only if its diagonal generator is either a substring of $(P_{0,2^v})^\infty$ (Class 0, 1, or 2) or an $(n-1)$-long substring of $0^{s-1}10^{u2^v-1}10^{2^v-1}$ containing at least two 1s (Class 3).*

Theorem 8 (Number of 1s on diagonal generators) *For $u \in \mathbb{O}_3$, $s \in \mathbb{Z}^+$, and $v \in \mathbb{N}$, a diagonal generator of a bipartite Steinhaus graph has*

- *no 1s for Class 0,*
- *exactly one 1 for Class 1,*
- *two or more 1s separated by $(2^v - 1)$ 0s for Class 2,*
- *exactly two 1s separated by $(u2^v - 1)$ 0s for Class 3.*

Theorem 9 (Forbidden substrings in diagonal generators) *Let $v \in \mathbb{N}$ and $u \in \mathbb{O}_1$. A Steinhaus graph is bipartite if and only if the diagonal generator does not contain a substring of the form $0^{2^v}10^{u2^v-1}1$ or $10^{u2^v-1}10^{2^v}$.*

Theorem 10 (Characterization by adjacency lists) *If $v \in \mathbb{N}$, $u \in \mathbb{O}_3$, and $s = \text{minAdj}(0)$, then a Steinhaus graph (except for the null graph) is bipartite if and only if either*

- $\text{Adj}^+(s) = \varnothing$ *(Class 1),*

- $\mathrm{Adj}^+(s) = \{s + 2^v\}$ *where* $S \leq 2^v$ *(Class 2), or*
- $\mathrm{Adj}^+(s) = \{s + u2^v\}$ *where* $S \leq 2^v$ *and* $n - s - u2^v \leq 2^v$ *(Class 3).*

4 The size of each class

In this section the size of Classes 0–2 are found and a recursion is given for the size of Class 3, along with two of its bizarre closed forms. Let $b(n)$ be the number of bipartite Steinhaus graphs with n vertices and $b_k(n)$ be the number of such graphs in Class k. The recursion for $b(n)$ is given in (4). We show that $b(n) = 2n - 2 + b_3(n)$ where $b_3(n)$ also satisfies (4).

Theorem 11 (The size of Class 1) *There are* $n - 1$ *graphs in Class 1.*

PROOF. Fixing s determines an alternative generator for Class 1 and since there are $n-1$ choices for s, there are $n-1$ graphs described by the alternative generators. Note also that for Class 1, there are $n-1$ positions on the diagonal generator to place the 1 and so there are $n-1$ graphs described by the diagonal generators. Likewise, there are $n-1$ vertices to choose that could be adjacent to vertex 0 and so the adjacency lists also describe $n-1$ graphs. $\square$

Theorem 12 (The size of Class 2) *There are* $n - 2$ *graphs in Class 2.*

PROOF. Let μ be the integer such that $2^\mu < n \leq 2^{\mu+1}$. By Theorem 5, the alternative generators of the Class 2 graphs can be described by choosing r such that $1 \leq r \leq \mu$ and further choosing $1 \leq s \leq 2^r$. If $r = \mu$ and s is too large, then this describes a graph from Class 1. To avoid this, we need $n = s + S2^v + t$ where $t > 0$ and $S2^v = 2^r$ (from the known generator description). So if $r = \mu$, then $s \leq n - 2^r + 1$. Hence there are

$$n - 2^\mu - 1 + \sum_{k=0}^{\mu-1} 2^k = n - 2^\mu - 1 + 2^\mu - 1 = n - 2 \tag{5}$$

generators of Class 2 graphs.

To count the diagonal generators directly, note that a diagonal generator has the form $0^{s-1}(P_{0,2^r})^\infty$ where $1 \leq s \leq 2^r$. Given 2^r, there are 2^r diagonal generators with period 2^r unless $r = \mu$ and then there are $n - 2^r - 1$ generators. Hence the total is the left-hand-side of (5).

The number of Class 2 graphs also can be counted by considering the number of adjacency lists. This is the number of ordered pairs (s, x) such that $s + x \leq n-1$ and $x = 2^v$, where v is a non-negative integer such that $1 \leq s \leq 2^v$. Note that for any integer i with $2 \le i \le n-1$, there is exactly one pair (s,x) with $s+x=i$ satisfying the above property; i.e., $x = 2^{\lfloor \lg(i-1) \rfloor}$ and $s = i-x$. Therefore, there are $n-2$ graphs in Class 2. $\square$

Theorem 13 (Recursion for the size of Class 3) *A recursion for $b_3(n)$ is $b_3(2)=0$, $b_3(3)=0$, and for $k \ge 2$,*

$$b_3(2k+1) = 2b_3(k+1)+1 \quad \textit{and} \quad b_3(2k) = b_3(k)+b_3(k+1). \tag{6}$$

PROOF. Since $b(n)$ satisfies (4) and $b_3(n) = b(n)-2n+2$, $b_3(n)$ also satisfies (4). $\square$

Summary 1 *Class 0 contains 1 graph, Class 1 contains $n-1$ graphs, and Class 2 contains $n-2$ graphs. Thus Class 3 contains $b(n)-2n+2$ graphs.*

We now give two quite different expressions for $b_3(n)$. First, we introduce the functions from $\mathbb{R}$ into $\mathbb{R}^+$ defined by

$$\mathrm{tooth}_v(x) = \begin{cases} 2^v, & \text{if } x \equiv 0 \pmod{2^{v+1}}; \\ 0, & \text{if } x \equiv 2^v \pmod{2^{v+1}}; \end{cases}$$

and by linear interpolation between these values. We can also write these functions as

$$\mathrm{tooth}_v(x) = \left| (x \bmod 2^{v+1}) - 2^v \right|.$$

Theorem 14 (Expression for $b(n)$: tooth version) *For $n > 2$,*

$$b(n) = 2n-2+\sum_{v=0}^{\left\lfloor \lg\left(\frac{n-1}{3}\right) \right\rfloor} \mathrm{tooth}_v(n-1).$$

PROOF. From Summary 1, we only need to compute $b_3(n)$. For those bipartite Steinhaus graphs which are not subgraphs of infinite bipartite Steinhaus graphs, each is diagonally generated by a sequence of the form $0^s10^w10^t$ for some non-negative integers s, t, u, v, and w with $w = u2^v-1$, $u \in \mathbb{O}_3$, $v \ge 1$ and $s,t < 2^v$. (Those graphs with $u=1$ are already counted since they are subgraphs of infinite Steinhaus graphs.) We count the number of such sequences for given values of u and v. As before, let $n_1 = n-1$ be the length of such a sequence. Then $n_1 = u2^v+1+s+t$ and $s+t = (n_1-1-2^v) \bmod 2^{v+1}$. For a given sum $x = s+t$, we distinguish two cases.

First, if $0 \le x < 2^v$, then we have $x+1$ sequences with $0 \le s \le x$ and $0 \le t = x-s \le x$. Second, if $2^v \le x < 2^{v+1}$, then we have $2^{v+1}-1-x$ sequences with $x-2^v+1 \le s \le 2^v-1$ and $x-2^v+1 \le t = x-s \le 2^v-1$.

Hence there are $\mathrm{tooth}_v(n_1)$ sequences of length n_1 for a given value of v with $0 \le v \le \lg\lfloor \frac{n_1}{3} \rfloor$. $\square$

Theorem 15 (Expression for $b(n)$: binary expansion of $n-2$ version) *Let $n_2 = (a_k a_{k-1} \ldots a_0)_2$ be the binary expansion of $n-2$ with $a_k = 1$. If c_i (respectively, d_i) is the number of 00 (respectively, 11) in $a_{k-1}a_{k-2}\ldots a_{i+1}$, then*

$$b(n) = 2n - 2 + a_0 + (a_{k-1}a_{k-2}\ldots a_1)_2 + \sum_{i=0}^{k-1}(c_i(1-a_i) + d_i a_i)2^i + d_{-1}. \quad (7)$$

PROOF. Again, we only need to compute $b_3(n)$ to prove (7). Denote by B_n the set of all (s, x') which satisfies the adjacency list conditions in Theorem 6; i.e., $x = s + u2^v$, $u \in \mathbb{O}_3$, $1 \le s \le 2^v$, $0 < n - x \le 2^v$, and $x' = u2^v$. Note that $b_3(n) = |B_n|$. Let $m = 2^{\lfloor \lg(n_2) \rfloor}$ (recall that $n_2 = n - 2$). For any $(s, x') \in B_n$, we claim that $x' \ge \frac{3}{4}m$. Otherwise, suppose $x' < \frac{3}{4}m$. If f is defined by (3) in the notation section, then $f(x') = f(u2^v) = 2^v$ and since $2^v \le \frac{1}{8}m$,

$$s \le 2^{\lceil \lg(s) \rceil} \le 2^v \le \tfrac{1}{8}m.$$

Hence

$$\tfrac{1}{8}m \ge 2^v \ge n - s - x' > n - \tfrac{1}{8}m - \tfrac{3}{4}m > \tfrac{1}{8}m,$$

a contradiction.

Since $2m > x' \ge \frac{3}{4}m$ and $x' \ne m$, $x' - \frac{1}{2}m$ is not a power of 2 except when $x' = \frac{3}{4}m$ or $x' = \frac{3}{2}m$. So if $x' \notin \{\frac{3}{4}m, \frac{3}{2}m\}$, then $2^v = f(x' - \frac{1}{2}m)$ and $2^v \ge n - x$ if and only if $f(x' - \frac{1}{2}m) \ge (n - \frac{1}{2}m) - s - (x' - \frac{1}{2}m)$. Hence $(s, x') \in B_n$ if and only if $(s, x' - \frac{1}{2}m) \in B_{n-\frac{1}{2}m}$, where $x' \notin \{\frac{3}{4}m, \frac{3}{2}m\}$. When $x' = \frac{3}{4}m$, $f(x') = \frac{1}{4}m$; and so $(s, x') \in B_n$ if and only if $0 < n - s - \frac{3}{4}m \le \frac{1}{4}m$ and $1 \le s \le \frac{1}{4}m$; i.e., $\frac{1}{4}m \ge s \ge n - m = n_2 - m + 2$. There are $\frac{1}{4}m - (n_2 - m) - 1$ (respectively, 0) s satisfying the above inequalities when $m \le n_2 < \frac{5}{4}m$ (respectively, otherwise). When $x' = \frac{3}{2}m$, $f(x') = \frac{1}{2}m$; and so $(s, x') \in B_n$ if and only if $0 < n - s - \frac{3}{2}m \le \frac{1}{2}m$ and $1 \le s \le \frac{1}{2}m$; i.e., $n_2 - \frac{3}{2}m + 1 \ge s \ge 1$. There are $n_2 - \frac{3}{2}m + 1$ (respectively, 0) s satisfying the above inequalities when $\frac{3}{2}m \le n_2 < 2m$ (respectively, otherwise). Thus

$$b_3(n) = b_3(n - \tfrac{m}{2}) + \begin{cases} \frac{m}{4} - (n_2 - m) - 1, & \text{if } m \le n_2 < \frac{5}{4}m; \\ 0, & \text{if } \frac{5}{4}m \le n_2 < \frac{3}{2}m; \\ n_2 - \frac{3}{2}m + 1, & \text{if } \frac{3}{2}m \le n_2 < 2m. \end{cases}$$

Then

$$b_3(n) = \begin{cases} b_3(n - \frac{m}{2}) \ + & \begin{cases} \frac{m}{4} - (n_2 - m) - 1, & \text{if } m \le n_2 < \frac{5}{4}m; \\ 0, & \text{if } \frac{5}{4}m \le n_2 < \frac{3}{2}m; \end{cases} \\ b_3(n - m) \ + & \begin{cases} \frac{m}{4}, & \text{if } \frac{3}{2}m \le n_2 < \frac{7}{4}m; \\ n_2 - \frac{3}{2}m + 1, & \text{if } \frac{7}{4}m \le n_2 < 2m. \end{cases} \end{cases} \tag{8}$$

Let $(a_k a_{k-1} \dots a_0)_2$ be the binary representation of n_2 with $a_k = 1$ and if a_i is a bit, then $\overline{a_i} = 1 - a_i$. For $n \ge 4$, (8) is the same as

$$\begin{aligned} b_3((1a_{k-1} \dots a_0)_2 + 2) &= b_3((1a_{k-2} \dots a_0)_2 + 2) \\ &+ \begin{cases} (\overline{a_{k-3} a_{k-4} \dots a_0})_2, & \text{if } a_{k-1}a_{k-2} = 00; \\ 0, & \text{if } a_{k-1}a_{k-2} = 01; \\ 2^{k-2}, & \text{if } a_{k-1}a_{k-2} = 10; \\ 2^{k-2} + (a_{k-3}a_{k-4} \dots a_0)_2 + 1, & \text{if } a_{k-1}a_{k-2} = 11. \end{cases} \end{aligned} \tag{9}$$

Also, $b_3((10)_2 + 2) = 0$ and $b_3((11)_2 + 2) = 1$. Repeatedly applying (9) gives

$$b_3(n) = a_0 + \sum_{\substack{a_{i-1}=1 \\ 2 \le i \le k}} 2^{i-2} + \sum_{i=0}^{k-1} c_i (1 - a_i) 2^i + \sum_{i=0}^{k-1} d_i a_i 2^i + d_{-1}$$

which can be rewritten as (7). □

5 Lower bound for $b(n)$

To find a lower bound for $b(n)$, consider formula (7). Let m be a positive integer for which the binary expansion of $m_2 = m - 2$ is a substring of $(10)^k 0$. Then in (7), $c_i = d_i = 0$ for all i. Hence, if $m_2 = (10)^k 0$, then $b_3(m) = (10)_2^{k-1} = \frac{2}{3}(4^k - 1)$, $m = \frac{1}{3}(4^{k+1} + 2)$, and $b_3(m) = \frac{1}{8}(m - 6)$. We now show that $\ell(n) = \frac{1}{8}(n - 6)$ is a lower bound for $b_3(n)$ for all positive integers n.

Lemma 16 (Lower bound for $b_3(n)$) *A lower bound for $b_3(n)$ is*

$$b_3(n) \ge \tfrac{1}{8}(n - 6). \tag{10}$$

PROOF. We induct on n. By inspection, the lemma is true for $n < 32$. Express n as $16q + r$, $0 \le r \le 15$, and consider four cases: r odd; $r \in \{0, 2, 4, 6, 12, 14\}$; q even, $r \in \{8, 10\}$; and q odd, $r \in \{8, 10\}$. Since $b_3(8q + k)$

is an integer, if $b_3(8q+k) \geq \ell(8q+k) = q + \frac{1}{8}(k-6)$, then

$$b_3(8q+k) \geq \begin{cases} q, & \text{if } 0 \leq k \leq 6; \\ q+1, & \text{if } k = 7. \end{cases} \tag{11}$$

Case 1 ($\boldsymbol{n = 16q + r,\ r}$ odd) Let $r = 2k+1$, $0 \leq k \leq 7$. Using (6) and (11), we have

$$b_3(16q+2k+1) = 2b_3(8q+k+1)+1 \geq \begin{cases} 2q+1, & \text{if } 0 \leq k \leq 6; \\ 2q+2, & \text{if } k = 7. \end{cases} \tag{12}$$

Now

$$\ell(16q+2k+1) = \tfrac{1}{8}(16q+2k-5) \leq \begin{cases} 2q+1, & \text{if } 0 \leq k \leq 6; \\ 2q+2, & \text{if } k = 7; \end{cases}$$

and hence $b_3(16q+2k+1) \geq \ell(16q+2k+1)$ for $0 \leq k \leq 7$.

Case 2 ($\boldsymbol{n = 16q + r,\ r \in \{0, 2, 4, 6, 12, 14\}}$) Let $r = 2k$ with $k \in \{0, 1, 2, 3, 6, 7\}$. Using (6) and (11), we have

$$\begin{aligned} b_3(16q+2k+1) &= b_3(8q+k) + b_3(8q+k+1) + 1 \\ &\geq \begin{cases} 2q, & \text{if } 0 \leq k \leq 5; \\ 2q+1, & \text{if } k = 6; \\ 2q+2, & \text{if } k = 7. \end{cases} \end{aligned} \tag{13}$$

Now

$$\ell(16q+2k) = \tfrac{1}{8}(16q+2k-6) \leq \begin{cases} 2q, & \text{if } 0 \leq k \leq 3; \\ 2q+1, & \text{if } 4 \leq k \leq 7; \end{cases}$$

and hence $b_3(16q+2k) \geq \ell(16q+2k)$ for $k \in \{0, 1, 2, 3, 6, 7\}$.

Before proceeding to Cases 3 and 4, note that

$$\ell(32q+2k) = \tfrac{1}{8}(32q+2k-6) \leq \begin{cases} 4q+1, & \text{if } 4 \leq k \leq 5; \\ 4q+3, & \text{if } 12 \leq k \leq 13. \end{cases} \tag{14}$$

Case 3 ($\boldsymbol{n = 16q + r,\ q}$ even, $\boldsymbol{r \in \{8, 10\}}$) So $n = 32q + 2k$, $4 \leq k \leq 5$. Using (6), (12), (13), and noting that one of k and $k+1$ is odd, gives

$$b_3(32q+2k) = b_3(16q+k) + b_3(16q+k+1) \geq 2q + (2q+1) = 4q+1.$$

By (14), $b_3(32q+2k) \geq \ell(32q+2k)$ for $4 \leq k \leq 5$.

Case 4 ($n = 16q+r$, q odd, $r \in \{8, 10\}$) So $n = 32q+16+2k$, $4 \le k \le 5$. Using (6), (12), (13), and noting that $8 + k$ is either 12 or 13, gives

$$\begin{aligned} b_3(32q + 16 + 2k) &= b_3(16q + 8 + k) + b_3(16q + 8 + k + 1) \\ &\ge (2q + 1) + (2q + 2) \\ &= 4q + 3. \end{aligned}$$

By (14), $b_3(32q + 16 + 2k) \ge \ell(32q + 16 + 2k)$ for $4 \le k \le 5$. $\square$

Theorem 17 (Tight bounds for $b(n)$) *For $n \ge 4$,*

$$\tfrac{1}{8}(17n - 22) \le b(n) \le \tfrac{1}{2}(5n - 7).$$

The lower bound is achieved for $n = \frac{1}{3}(4^{k+1} + 2)$ and the upper bound is achieved for $n = 2^k + 1$.

PROOF. The upper bound has already been given and since $\frac{1}{8}(17n - 22) = 2n - 2 + \frac{1}{8}(n - 6)$ and $b(n) = 2n - 2 + b_3(n)$, the inequality follows from Lemma 16. In the introduction to this section, we showed that the lower bound is achieved for $n = \frac{1}{3}(4^{k+1} + 2)$. $\square$

Since $b(n)$ is an integer for all n, we can replace the lower and upper bounds in the previous theorem with the ceiling and floor, respectively.

Theorem 18 (Tighter bounds for $b(n)$) *For $n \ge 4$,*

$$\left\lceil \tfrac{1}{8}(17n - 22) \right\rceil \le b(n) \le \left\lfloor \tfrac{1}{2}(5n - 7) \right\rfloor.$$

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