procedure PRIM_MST (V, E, w, r)

$V_T := \{r\}$
$d[r] := 0$
for all $v \in (V - V_T)$ do
  if edge (r, v) exists set $d[v] := w(r, v)$
  else set $d[v] := \infty$
while $V_T \neq V$ do
  find vertex u such that $d[u] = \min\{d[v] \mid v \in (V - V_T)\}$
  $V_T := V_T \cup \{u\}$
  for all $v \in (V - V_T)$ do
    $d[v] = \min\{d[v], w(u, v)\}$

Dijkstra's single-source shortest paths algorithm is identical to Prim's MST algo, except for each $v \in (V - V_T)$

→ Dijkstra's stores $l[u]$, the min cost to reach u from s via V_T
  ie $l[v] = \min\{l[v], l[u] + w(u,v)\}$ is the update step

→ Prim's stores $d[u]$, cost of min-cost edge connecting vertex in V_T to u

• Parallel formulation of both algorithms are the same

• Dijkstra's all-pairs shortest paths : $\Theta(n^3)$

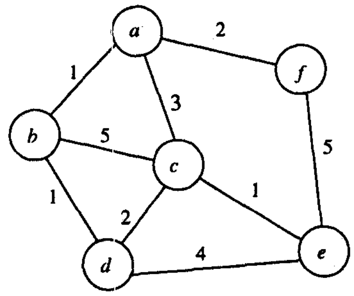
→ Source-partitioned parallel formulation : uses n processors only
  $T_p = \Theta(n^2)$; $S = \Theta(n^3)/\Theta(n^2)$; $E = \Theta(1)$; Isoefficiency$_{comm} = \Theta(p^3)$
  very poor scalability

→ Use source-parallel formulation : p $(>n)$ processors
  $\frac{p}{n}$ procs / partition ; each partition runs the 1-source shortest path algo

Can use $\Theta(n^2)$ processors efficiently

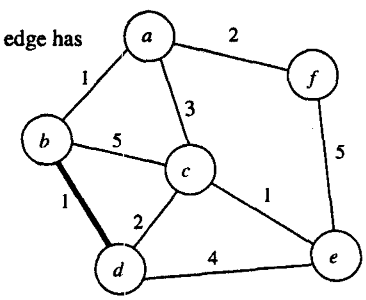
• $G(V, E, w, r)$

(a) Original graph



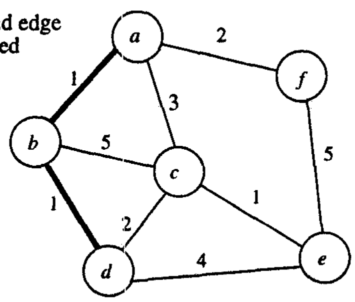
	a	b	c	d	e	f
d[]	1	0	5	1	∞	∞
a	0	1	3	∞	∞	2
b	1	0	5	1	∞	∞
c	3	5	0	2	1	∞
d	∞	1	2	0	4	∞
e	∞	∞	1	4	0	5
f	2	∞	∞	∞	5	0

(b) After the first edge has been selected



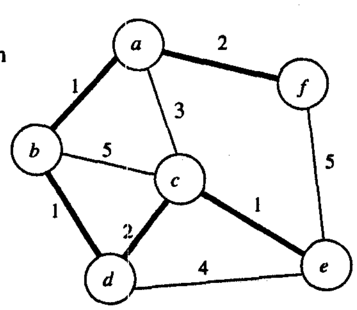
	a	b	c	d	e	f
d[]	1	0	2	1	4	∞
a	0	1	3	∞	∞	2
b	1	0	5	1	∞	∞
c	3	5	0	2	1	∞
d	∞	1	2	0	4	∞
e	∞	∞	1	4	0	5
f	2	∞	∞	∞	5	0

(c) After the second edge has been selected



	a	b	c	d	e	f
d[]	1	0	2	1	4	2
a	0	1	3	∞	∞	2
b	1	0	5	1	∞	∞
c	3	5	0	2	1	∞
d	∞	1	2	0	4	∞
e	∞	∞	1	4	0	5
f	2	∞	∞	∞	5	0

(d) Final minimum spanning tree



	a	b	c	d	e	f
d[]	1	0	2	1	1	2
a	0	1	3	∞	∞	2
b	1	0	5	1	∞	∞
c	3	5	0	2	1	∞
d	∞	1	2	0	4	∞
e	∞	∞	1	4	0	5
f	2	∞	∞	∞	5	0

Figure 7.5 Prim's minimum spanning tree algorithm. The MST is rooted at vertex b. During each iteration, the minimum cost edge connecting a vertex in V_T to a vertex in $V - V_T$ is selected and the corresponding vertex is added to V_T (shown shaded in the distance array d). The $d[v]$ values of the vertices in $V - V_T$ are then updated.

• $\Theta(n^2)$

Parallel Formulation : block-striped partitioning

- In each iteration, P_i computes $d_i[u] = \min\{d_i[v] \mid v \in (V - V_T) \cap V_i\}$
- Then do single-node accumulation at P_0 to find global minimum $d_i[u]$
- P_0 does $1 \to$ all BC of u
- Each P_i updates values of $d[v]$ for its local vertices
- (Each P_i needs to store the columns of the weighted adjacency matrix for V_i assigned to it.)
- Space $= \Theta(n^2/p)$; computation in each iteration $= \Theta(n/p)$

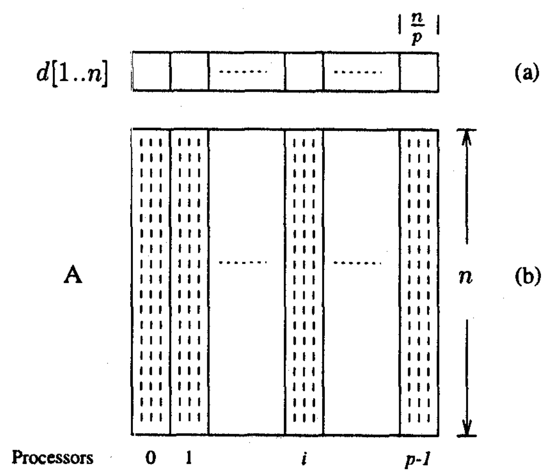


Figure 7.6 The partitioning of the distance array d and the adjacency matrix A among p processors. Copyright (r) 1994 Benjamin/Cummings Publishing Co.

HC: $T_p = \overbrace{\Theta\left(\frac{n^2}{p}\right)}^{\text{computation}} + \overbrace{\Theta(n \log p)}^{\text{communication}}$

$S = \dfrac{\Theta(n^2)}{\Theta(n^2/p) + \Theta(n \log p)}$

$E = \dfrac{1}{1 + \Theta((p \log p)/n)}$

For cost-optimality, $(p \log p)/n = O(1)$

Isoefficiency fn (due to communication) is $\Theta(p^2 \log^2 p)$

Mesh: $T_p = \Theta(n^2/p) + \Theta(n\sqrt{p})$

$S = \dfrac{\Theta(n^2)}{\Theta(n^2/p) + \Theta(n\sqrt{p})}$

$E = \dfrac{1}{1 + \Theta(p^{1.5}/n)}$

For cost-optimality, $p^{1.5}/n = O(1)$

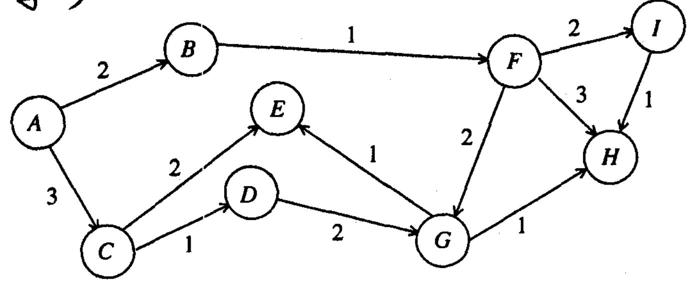
$\Rightarrow O(n^{2/3})$ processors can be used efficiently

Isoefficiency (due to communication) $= \Theta(p^3$

- $d_{ij}^{(k)} = \min_{1 \le m \le n} \{ d_{im}^{(k-1)} + w(v_m, v_j) \}$
- $D^{(k)}$ computed from $D^{(k-1)}$ & A using modified matrix multiplication

$c_{ij} = \min_{k=1}^{n} \{(a_{i,k} + b_{k,j})\}$, instead of $\sum_{k=1}^{n} a_{i,k} \times b_{k,j}$

- Total $\lceil \log(n-1) \rceil$ steps, ie, $A^2, A^4, A^8, \ldots$
- Complexity = $\Theta(n^3 \log n)$ $\Longrightarrow$ not optimal, but has a high degree of parallelism



$$A^1 = \begin{pmatrix} 0 & 2 & 3 & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty & \infty & 1 & \infty & \infty & \infty \\ \infty & \infty & 0 & 1 & 2 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & 0 & \infty & \infty & 2 & \infty & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty & 0 & 2 & 3 & 2 \\ \infty & \infty & \infty & \infty & 1 & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix} \quad A^2 = \begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 3 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty & \infty & 1 & 3 & 4 & 3 \\ \infty & \infty & 0 & 1 & 2 & \infty & 3 & \infty & \infty \\ \infty & \infty & \infty & 0 & 3 & \infty & 2 & 3 & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 3 & 0 & 2 & 3 & 2 \\ \infty & \infty & \infty & \infty & 1 & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 3 & 5 & 6 & 5 \\ \infty & 0 & \infty & \infty & 4 & 1 & 3 & 4 & 3 \\ \infty & \infty & 0 & 1 & 2 & \infty & 3 & 4 & \infty \\ \infty & \infty & \infty & 0 & 3 & \infty & 2 & 3 & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 3 & 0 & 2 & 3 & 2 \\ \infty & \infty & \infty & \infty & 1 & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix} \quad A^8 = \begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 3 & 5 & 6 & 5 \\ \infty & 0 & \infty & \infty & 4 & 1 & 3 & 4 & 3 \\ \infty & \infty & 0 & 1 & 2 & \infty & 3 & 4 & \infty \\ \infty & \infty & \infty & 0 & 3 & \infty & 2 & 3 & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 3 & 0 & 2 & 3 & 2 \\ \infty & \infty & \infty & \infty & 1 & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix}$$

Figure 7.7 An example of the matrix-multiplication-based all-pairs shortest paths algorithm.

- On a HC w/ n^3 processors, $n \times n$ matrix mult in $\Theta(\log n)$ [see MM chapt.]

$S = \frac{\Theta(n^3)}{\Theta(\log^2 n)}$; $E = \frac{1}{\Theta(\log^2 n)}$

Source-parallel formulation of Dijkstra's algorithm

→ single-source algo. on each submesh : $T_p = \underbrace{\Theta(n^3/p)}_{\text{computation}} + \underbrace{\Theta(\sqrt{np})}_{\text{communic}}$

$S = \dfrac{\Theta(n^3)}{\Theta(n^3/p) + \Theta(\sqrt{np})}$, $E = \dfrac{1}{1+\Theta(p^{1.5}/n^{2.5})}$

For cost-optimality, $p^{1.5}/n^{2.5} = O(1) \implies$ can use $O(n^{1.66})$ processors efficiently

$\text{Isoefficiency}_{\text{communic}} = \Theta(p^{1.8})$

$\text{Isoefficiency}_{\text{concurrency}} = \Theta(p^{1.5})$

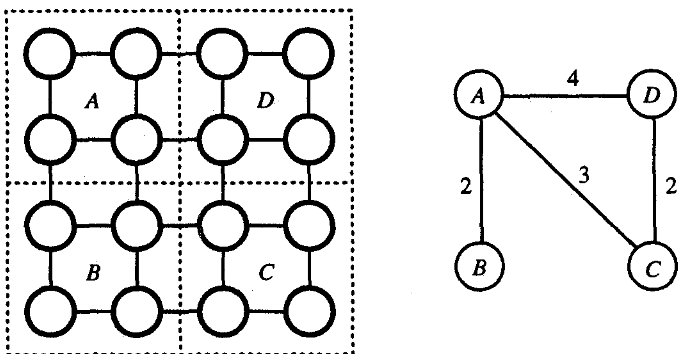


Figure 7.8 Partitioning a $\sqrt{p} \times \sqrt{p}$ mesh into n submeshes, each of size $\sqrt{p/n} \times \sqrt{p/n}$. In this example, $p = 16$ and $n = 4$. Each of the $\sqrt{p/n} \times \sqrt{p/n} = 2 \times 2$ meshes solves the single-source shortest paths problem for a given source vertex. In this example, each submesh is marked by the source vertex of its single-source algorithm.

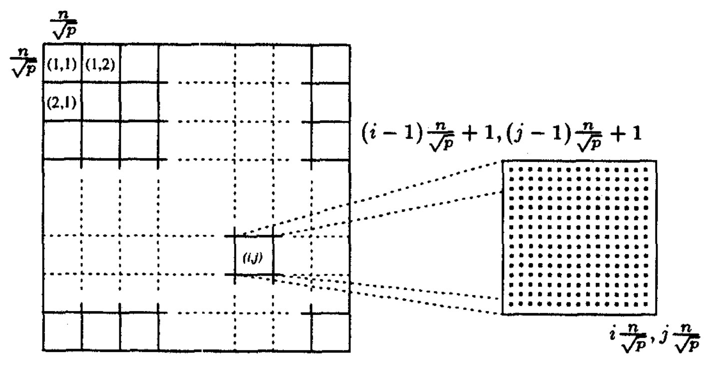
FLOYD's algorithm

$$d_{i,j}^{(k)} = \begin{cases} w(v_i, v_j) & \text{if } k = \phi \\ \min\left\{ d_{i,j}^{(k-1)},\ d_{i,k}^{(k-1)} + d_{k,j}^{(k-1)} \right\} & \text{if } k \geq 1 \end{cases}$$

- Solve bottom-up in the order of increasing values of k ; $\theta(n^3)$

space $= \theta(n^2)$

$D^{(0)} = A$
for $k = 1$ to n do
  for $i = 1$ to n do
    for $j = 1$ to n do
      $d_{i,j}^{(k)} := \min\left(d_{i,j}^{(k-1)},\ d_{i,k}^{(k-1)} + d_{k,j}^{(k-1)} \right)$



(a) (b)

Figure 7.9 (a) Matrix $D^{(k)}$ partitioned by block checkerboarding into $\sqrt{p} \times \sqrt{p}$ subblocks, and (b) the square subblock of $D^{(k)}$ assigned to processor $P_{i,j}$.

Total of p blocks

Parallel formulations: to compute portion of $D^{(k)}$, need corresp. segments of k^{th} row and column of $D^{(k-1)}$

- k^{th} iteration: each of the $\sqrt{p}$ processors containing part of the k^{th} row (column) send it to the $\sqrt{p}-1$ processors in the same column (row)

- embed mesh into p-proc HC with cut-through routing
  - → $\sqrt{p} \times \sqrt{p}$ virtual mesh $\longleftrightarrow$ p-proc HC
  - → each row or column of mesh $\equiv$ HC of $\sqrt{p}$ procs
  - → in each iteration, k^{th} row & k^{th} column do $1 \to$ all BC along a column or row

    (each such proc has $n/\sqrt{p}$ elements of k^{th} row or col)

    $\Longrightarrow$ BC requires $\Theta((n \log p)/\sqrt{p})$ time

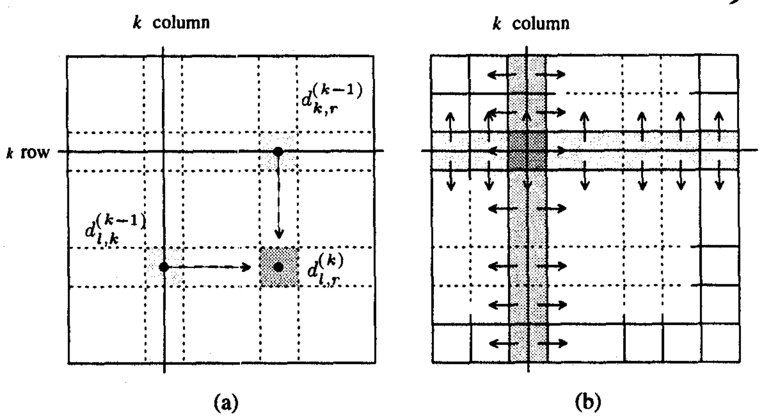


Figure 7.10 (a) Communication patterns used in the block-checkerboard partitioning. When computing $d_{i,j}^{(k)}$, information must be sent to the highlighted processor from two other processors along the same row and column. (b) The row and column of $\sqrt{p}$ processors that contain the k^{th} row and column send them along processor columns and rows.

→ $\Theta(n^2/p)$ to compute values of $D^{(k)}$ in block

$$T_p = \Theta(n^3/p) + \Theta\left(\frac{n^2}{\sqrt{p}} \log p\right)$$

$$S = \frac{\Theta(n^3)}{\Theta(n^3/p) + \Theta((n^2 \log p)/\sqrt{p})}$$

$$E = \frac{1}{1+\Theta((\sqrt{p} \log p)/n)}$$

For cost-optimality, $\sqrt{p} \log p / n = O(1)$

$\text{Isoefficiency}_{comm} = \Theta(p^{1.5} \log^3 p)$

$\text{Isoefficiency}_{\cancel{comput}\ concurrency} = \Theta(p^{1.5})$

Prim's algorithm using adjacency lists & binary heap: $\Theta(|E| \log n)$

$\rightarrow$ better if $|E| = O(n^2/\log n)$

Complexity of adjacency-list based algorithms $\Omega(n + |E|)$

- difficult to achieve even work distribution & low comm ovhd for random sparse graphs

∴ focus on grid-like graphs

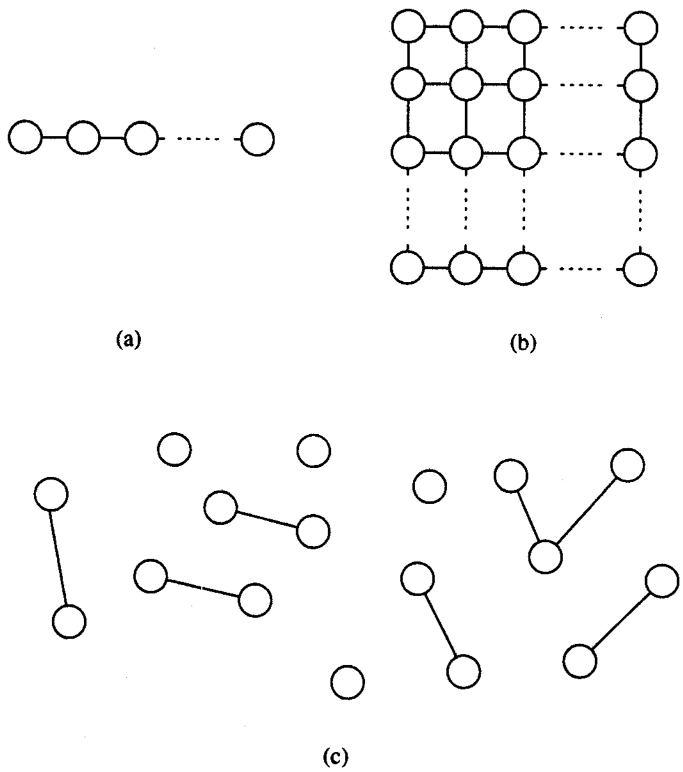
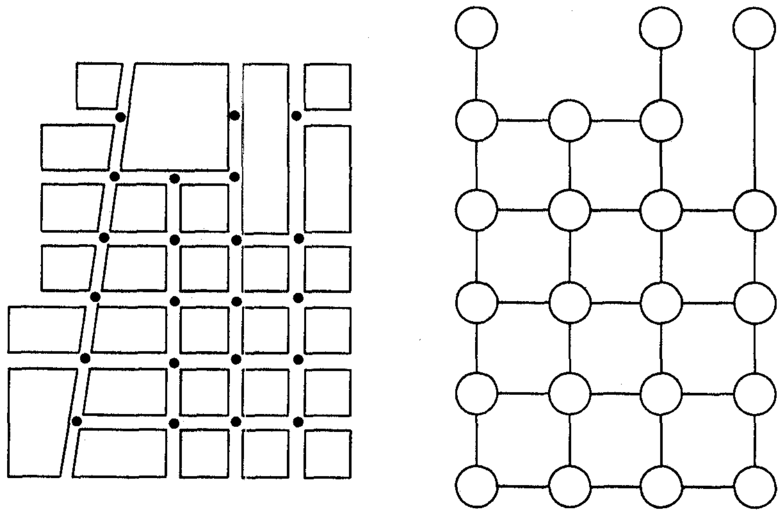


Figure 7.15 Examples of sparse graphs: (a) a linear graph, in which each vertex has two incident edges; (b) a grid graph, in which each vertex has four incident vertices; and (c) a random sparse graph.

Johnson_SSSP (V, E, s) /* modification of Dijkstra's SSSP */

```
Q := V
for all v ∈ Q do
    l[v] := ∞
l[s] := 0
while Q ≠ ∅ do
    u := extract_min(Q);
    for each v ∈ Adj[u] do
        if v ∈ Q and l[u] + w(u,v) < l[v] then
            l[v] := l[u] + w(u,v)
```



(a) (b)

Figure 7.16 A street map (a) can be represented by a graph (b). In the graph shown in (b), each street intersection is a vertex and each edge is a street segment. The vertices of (b) are the intersections of (a) marked by dots.

- Q implemented as a binary min-heap
  → each update in $O(\log n)$
- Complexity $= \Theta(|E| \log n)$
- To parallelize, distribute the priority queue

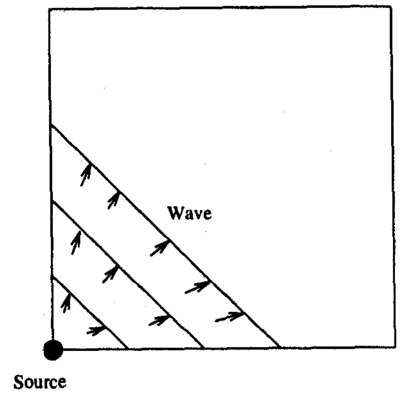


Figure 7.18 The wave of activity in the priority queues.

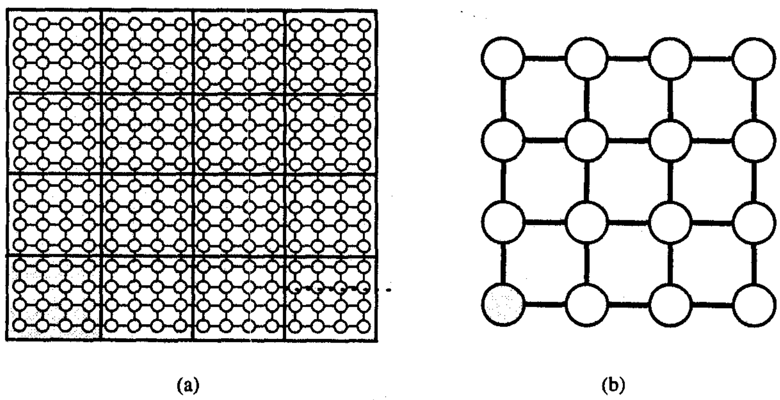
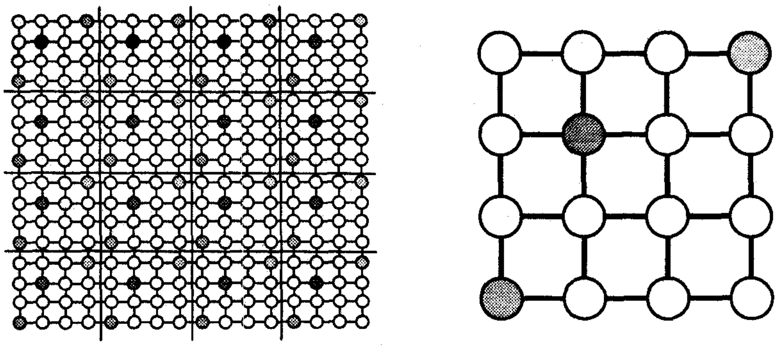


Figure 7.19 Mapping the grid graph (a) onto a mesh (b) by using the block-checkerboard mapping. In this example, $n = 16$ and $\sqrt{p} = 4$. The shaded vertices are mapped onto the shaded processor.

- assign $\frac{n}{\sqrt{p}} \times \frac{n}{\sqrt{p}}$ vertices to each processor, given $n \times n$ grid
- # busy processors = # processors intersected by wave
- Let W be overall work done by sequential algorithm

$S_{max} = \frac{W}{W/\sqrt{p}} = \sqrt{p}$; $E_{max} = 1/\sqrt{p}$

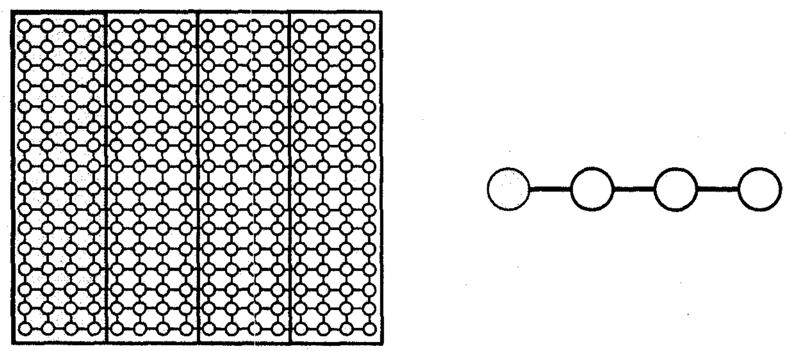
• each processor responsible for vertices that belong to different parts of the grid graph $\Rightarrow$ procs remain busy most of the time, but higher communic.



(a) (b)

Figure 7.20 Mapping the grid graph (a) onto a mesh (b) by using the cyclic-checkerboard mapping. In this example, $n = 16$ and $\sqrt{p} = 4$. The shaded graph vertices are mapped onto the correspondingly shaded mesh processors.

[as adjacent vertices are assigned to separate processors, more communic necessary each time P_i extracts a node from Q_i]



(a) (b)

Figure 7.21 Mapping the grid graph (a) onto a linear array of processors (b). In this example, $n = 16$ and $p = 4$. The shaded vertices are mapped onto the shaded processor.

Block-striped mapping: on avg, $\frac{p}{2}$ processors are busy

$S = \frac{p}{2}$; $E = \frac{1}{2}$. However, cannot use more than $O(n)$ procs

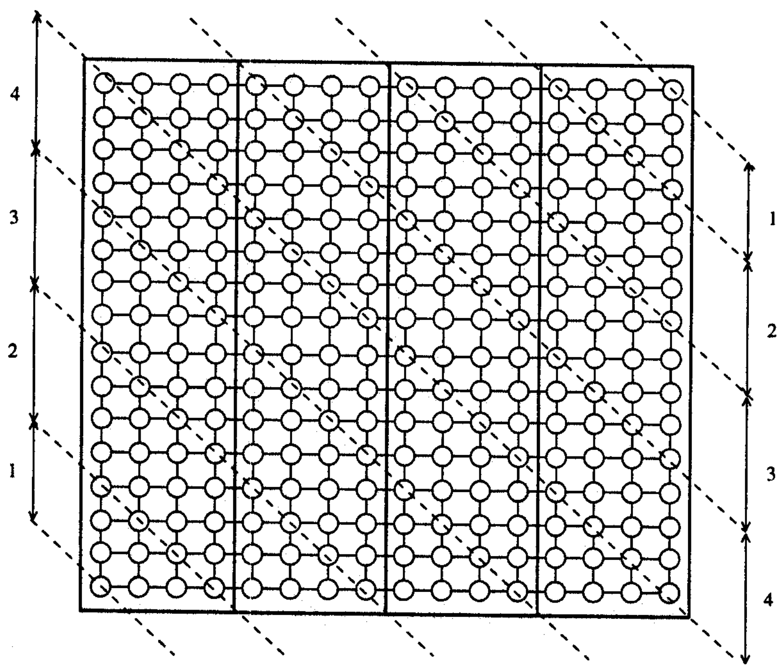


Figure 7.22 The number of busy processors as the computational wave propagates across the grid graph.