Math 535 Homework 2

Due February 1

1) Suppose $Q(z)$ is a polynomial with distinct complex roots $z_1, ..., z_n$, and let $P(z)$ be a polynomial of some degree $m < n$. Let $c_i = \frac{P(z_i)}{Q'(z_i)}$. Show that the partial fractions decomposition of $R(z) = \frac{P(z)}{Q(z)}$ is given by

$$R(z) = \sum_{i=1}^{n} \frac{c_i}{z - z_i}$$

2a) Suppose $\sum_n a_n z^n$ has radius of convergence R, where $0 < R < \infty$. What is the radius of convergence of $\sum_n a_n^2 z^n$?

2b) Show that for any increasing sequence $\{n_k\}$ the radius of convergence of $\sum_k a_{n_k} z^{n_k}$ is at least R. Give an example where it is strictly greater than R for at least one such sequence.

3a) Let $R(z)$ be a Möbius transformation $\frac{az+b}{cz+d}$. Write a formula for $R^{-1}(z)$, the inverse function to $R(z)$.

3b) Let z_1, z_2 and z_3 be distinct points in the complex plane. Find a Möbius transformation taking z_1 to 0, z_2 to 1, and z_3 to ∞.

4) Find all z such that $\sin(z)$ is real, and all z such that $\sin(z)$ is purely imaginary.

5) Suppose $R(z) = \frac{P(z)}{Q(z)}$ is a rational function of order d. What are the minimum and maximum possible orders of $R'(z)$? (Hint: Partial fractions.)

6) Let Ω be a domain. Suppose $u(x, y)$ is a function on Ω such that $u(x, y)$ and $(u(x, y))^2$ are both harmonic. Show that $u(x, y)$ is constant.