# Optimal Multichannel Communication under Failure (Extended Abstract)

Tanya Y. Berger-Wolf* Edward M. Reingold†

October 19, 1998

**Abstract**

We consider the problem of communication over multiple channels in which channels can fail and the information sent over those channels is lost. The goal is to recover as much of the original information as possible while having as little redundancy as possible. We analyze the problem of designing an encoding that minimizes the decoding error in case of channel failure; we give tight lower bounds and an optimal algorithm.

## 1 Problem Statement

Consider sending $\lg m$ bits of information over $k$ independent channels. We want to partition the $\lg m$ bits into $k$ subsets so that if all $k$ subsets are received, the original information can be completely reconstructed. A trivial solution would be to send $k$ blocks of $(\lg m)/k$ bits each; if any block fails to arrive, that part of the information is lost. Alternatively, we could send $k$ complete copies, a total of $k \times \lg m$ bits, so that even if all but one of the channels fail, all the information is still available; however, in this case, if more than one channel succeeds, there is a lot of redundancy.

We are given $k$ channels of capacities $\log n_1, \log n_2, ..., \log n_k$ bits. We consider the information to be transmitted as a number $M$ with at most $\lg m$ bits, $1 \leq m \leq n_1 n_2 \cdots n_k$. $M$ is mapped to a unique $k$-tuple $(i_1, ..., i_k)$ and component $i_j$ is sent over the channel $j$. If all of the channels succeed, then we should be able to decode $M$ from the $k$-tuple. If some of the channels fail, then the information we have about $M$ is that it is between the smallest and the largest numbers among all those in the encoding scheme with the components of the successful channels equal to those components of $M$. Thus the problem of designing a code to minimize the decoding error in case of channel failure in such a system is equivalent to the problem of arranging $m$ numbers in a $k$-dimensional $n_1 \times n_2 \times ... \times n_k$ matrix to minimize the *spread*—the difference between the minimum and the maximum number in each projection of the matrix.

## 2 Problem Background

The problem of designing codes for a *diversity-based* (multichannel) communication system that guarantee a bound on the error based on the number of failed channels is known as the *multiple descriptions problem* [3]. It is a generalization of the problem of source coding subject to a *fidelity criterion* [5]. The problem generally consists of two parts. First, mapping the information, generally a real number, to a set of integers (*designing a quantizer*), second, mapping each integer to the set of components to be transmitted over the channels (*index assignment*). This paper deals with the second part of the problem. The relevant works include [3], [4], [1], [12], [10], [11], [6], [8], [2]. Applications of this problem arise in video and speech communication.

To the best of our knowledge, this paper is first to give lower bounds for the problem, and provide optimal algorithms that achieve the lower bound. In addition, the general case of $k$ channels, when $k > 2$, has never been addressed. A solution for $k = 2$ proposed by Vaishampayan [7] arranges the numbers in a diagonal of thickness (maximum number of elements in a row or column) $b$. He considers only the case of equal channel capacity, $n_1 = n_2 = n$. When $b < n+1$ Vaishampayan's solution achieves the bound we prove. However, when $b \geq n + 1$ his solution is not optimal. Our algorithm achieves the optimal bound.

In practice, there is usually a probability distribution associated with the numbers that can be sent. However, Vaishampayan [7] shows that the spread is asymptotically a good error measure and thus we do not take

*Dept. of Comp. Sci., Univ. of Illinois at Urbana-Champaign, 1304 W. Springfield Avenue, Urbana, Illinois 61801, USA. Supported in part by an NSF Graduate Fellowship and by NSF grant CCR-95-30297. Email: tanyabw@uiuc.edu

†Dept. of Comp. Sci., Univ. of Illinois at Urbana-Champaign, 1304 W. Springfield Avenue, Urbana, Illinois 61801, USA. Supported in part by NSF grant CCR-95-30297. Email: reingold@cs.uiuc.edu

the probability into consideration. Nonetheless, having a probability distribution associated with the set of numbers to be arranged in a matrix is an interesting problem for investigation.

This problem is also equivalent to minimizing graph bandwidth of a $k$-fold cartesian product of cliques and induced subgraphs of it. For discussion on the problem of graph bandwidth see West [9].

## 3 Solution Overview

Given an encoding, which is an arrangement of $m$ numbers in a $n_1 \times \cdots \times n_k$ $k$-dimensional matrix, the error in case of failure of channels $f_1, f_2, ..., f_l$ is the difference between the minimum and the maximum numbers, the spread, in the $l$-dimensional projection of the successful channels onto the the space of the failed channels:

$$(i_1, ..., i_{f_1-1}, *, i_{f_1+1}, ..., i_{f_l-1}, *, i_{f_l+1}, ..., i_k),$$

where $i_j$ is the number transmitted and received over a successful channel. The idea of the proof of the lower bound on the spread is as follows:

1. For any possible arrangement of $m$ numbers in a $k$-dimensional matrix, consider the ascending sequence of the minima in all possible $l$-dimensional projections of the matrix. Find a sequence $a_j$ that is lexicographically greater or equal to any such sequence. Then the $j$th largest minimum in any arrangement of $m$ numbers in the matrix is at most the $j$th member of the $a_j$ sequence.

2. Find the corresponding sequence $b_j$ that is lexicographically less or equal to any ascending sequence of maxima of $l$-dimensional projections of the matrix.x

3. For any arrangement of $m$ numbers in the $k$-dimensional matrix, the spread in $l$-dimensional projections is at least the maximum difference $b_j - a_j$.

The key observation to finding the bounding sequence of minima and maxima is that if $x$ is in a cell that is an intersection of $\binom{k}{l}$ $l$-dimensional projections, $t$ of which already have numbers less than $x$, then $x$ will appear in the minima sequence exactly $\binom{k}{l} - t$ times. Thus to maximize the sequence of minima, one must put numbers into cells that are intersections of as many projections as possible that already have numbers smaller than the current one; and similarly for the sequence of maxima. This leads to an inductive proof that translates to a recursive algorithm, resulting in a generalized "herringbone" arrangement schematically shown for two dimensions in Figure 1.

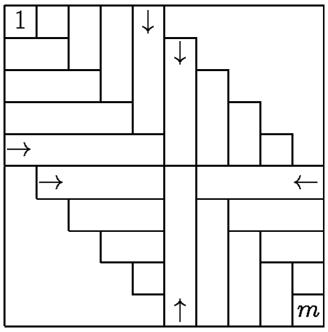


Figure 1: The "herringbone" arrangement of numbers in a matrix.

## References

[1] Berger, T. and Z. Zhang, "Minimum breakdown degradation in binary source encoding", *IEEE Transactions on Information Theory*, **29** (1983), 807–814.

[2] Buzi, L., V.A. Vaishampayan and R. Laroia, "Design and asymptotic performance of a structured multiple description vector quantizer", *Proc. IEEE Int. Symp. Inform. Theory*, 1994, IEEE, Piscataway, NJ, USA.

[3] El Gamal, A. A. and T.M. Cover, "Achievable rates for multiple descriptions", *IEEE Trans. Inform. Theory*, **28** (1982), 851–857.

[4] Ozarow, L., "On a source coding problem with two channels and three receivers", *Bell Syst. Tech. J.*, **59** (1980), 1909–1921.

[5] Shannon, C. E., "Coding theorems for discrete source with fidelity criterion", *IRE Nat. Conv. Rec.*, **4** (1959), 142–163.

[6] Vaishampayan, V. A., "Vector quantizer design for diversity systems", in *Proc. 25 Ann. Conf. Inform. Sci. Syst.*, Johns Hopkins University, Mar. 20–22, 1991, 564–569.

[7] Vaishampayan, V. A.,"Design of multiple description scalar quantizers," *IEEE Trans. Inform. Theory,* **93** (1993), 821–834.

[8] Vaishampayan, V. A. and J. Domaszewicz, "Design of entropy-constrained multiple-description scalar quantizers", *IEEE Trans. Inform. Theory*, **40** (1994), 245–250.

[9] West, D. B., *Introduction to Graph Theory*, Prentice Hall, 1996.

[10] Witsenhausen, H. S. and A.D. Wyner, "Source coding for multiple descriptions II: A binary source", *Bell Syst. Tech. J.*, **60** (1981), 2281–2292.

[11] Wolf, J. K., A.D. Wyner and J. Ziv, "Source coding for multiple descriptions", *Bell Syst. Tech. J.*, **59** (1980), 1417–1426.

[12] Zhang, Z. and T. Berger, "New results in binary multiple description", *IEEE Trans. Inform. Theory*, **33** (1987), 502–521.