Homework 5: Due 11:59pm Monday May 6, 2024

(Problem 5 is required for graduate students. Undergraduate students who solves Problem 5 can earn at most 5 points extra credit.)

Problem 1: (25 points)
The following statements are correct or not. No justification needed.

1. Weighted Interval Scheduling $\leq_P$ Interval Scheduling.
2. Independent Set $\leq_P$ Interval Scheduling.
3. If a computational problem is not in P, then it is in NP.
4. If P = NP, then all the computational problems can be solved in polynomial time.
5. If Vertex Cover $\leq_P$ Independent Set, then P = NP.

Problem 2: (24 points)
We define the decision version of the Shortest Path problem as follows: given an undirected graph $G = (V, E)$, two distinct vertices $s, t \in V$, and a positive integer k, return yes if and only if there exists a path from s to t of length at most k. Otherwise, return no.
For each of the two questions below, decide whether the answer is (i) "Yes", (ii) "No", or (iii) "Unknown, because it would resolve the question of whether P=NP." Give an explanation of your answers.

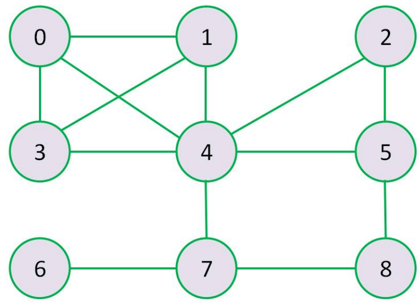
1. Question: (12 points) Is it the case that Shortest Path $\leq_P$ Independent Set?
2. Question: (12 points) Is it the case that Independent Set $\leq_P$ Shortest Path?

Problem 3: (24 points)
We say a graph $G = (V, E)$ has a k-coloring for some positive integer k if we can assign k different colors to vertices of G such that for every edge $(v, w) \in E$, the color of v is different to the color w. More formally, $G = (V, E)$ has a k-coloring if there is a function $f : V \to \{1, 2, \ldots, k\}$ such that for every $(v, w) \in E$, $f(v) \neq f(w)$.

1. (12 points) Is k-coloring problem a problem in NP? Why?
2. (12 points) 3-Color problem is defined as follows: Given a graph $G = (V, E)$, does it have a 3-coloring? 4-Color problem is defined as follows: Given a graph $G = (V, E)$, does it have a 4-coloring?
   Show that 3-Color $\leq_P$ 4-Color.

   (Hint: For any input of the 3-Color problem, add an auxiliary vertex, and properly add edges from the auxiliary vertex to other vertices.)

Problem 4: (27 points)
Complete Subgraph problem is defined as follows: Given a graph $G = (V, E)$ and an integer k, output yes if and only if there is a subset of vertices $S \subseteq V$ such that $|S| = k$, and every pair of vertices in S are adjacent (there is an edge between any pair of vertices).



For example, for the following graph and $k = 4$, the answer is yes, because $S = \{0, 1, 3, 4\}$ satisfies the requirement. But if $k \geq 5$, the answer is no.

1. (13 points) Show that the Complete Subgraph problem is in NP. Define your certificate, and describe the certificate algorithm using the certificate you defined.

2. (14 points) Show that Complete Subgraph problem is NP-Complete.

   (hint 1: the **Independent Set** problem is a NP-Complete problem.)

   (hint 2: You can also use other NP-Complete problems to prove NP-Complete of Complete Subgraph.)

Problem 5: (25 points, 5 bonus points for undergraduate students)
The ***Number Partition*** problem asks, given a collection of non-negative integers $S = \{x_1, \ldots, x_n\}$ whether or not the integers can be partitioned into two sets A and $\overline{A} = S - A$ such that

$$\sum_{x \in A} x = \sum_{x \in \overline{A}} x.$$

1. Show that *Number Partition* is in NP. Define your certificate, and describe the certificate algorithm using the certificate you defined.

2. Show that *Number Partition* is NP-Complete. Describe your reduction. **Make sure the direction of reduction is correct!**

   (hint 1: It is known that the ***Subset Sum*** problem is a NP-Complete problem. In the *Subset Sum* problem, we are given a collection of non-negative integers $Y = \{y_1, y_2, \ldots, y_m\}$ and a target integer t, we want to see if it is possible find a subset $Z \subseteq Y$ such that the sum of integers in Z equals to t

   $$\sum_{x \in Z} x = t.$$

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   (hint 2: Add an integer for the reduction.)