CS 401: Computer Algorithm I

Interval Partitioning / Lateness Minimization

Xiaorui Sun

Greedy Algorithm

- Greedy algorithm:
  - Divide solution construction into small steps
  - Decision in each step: ‘best’ current partial solution at each step

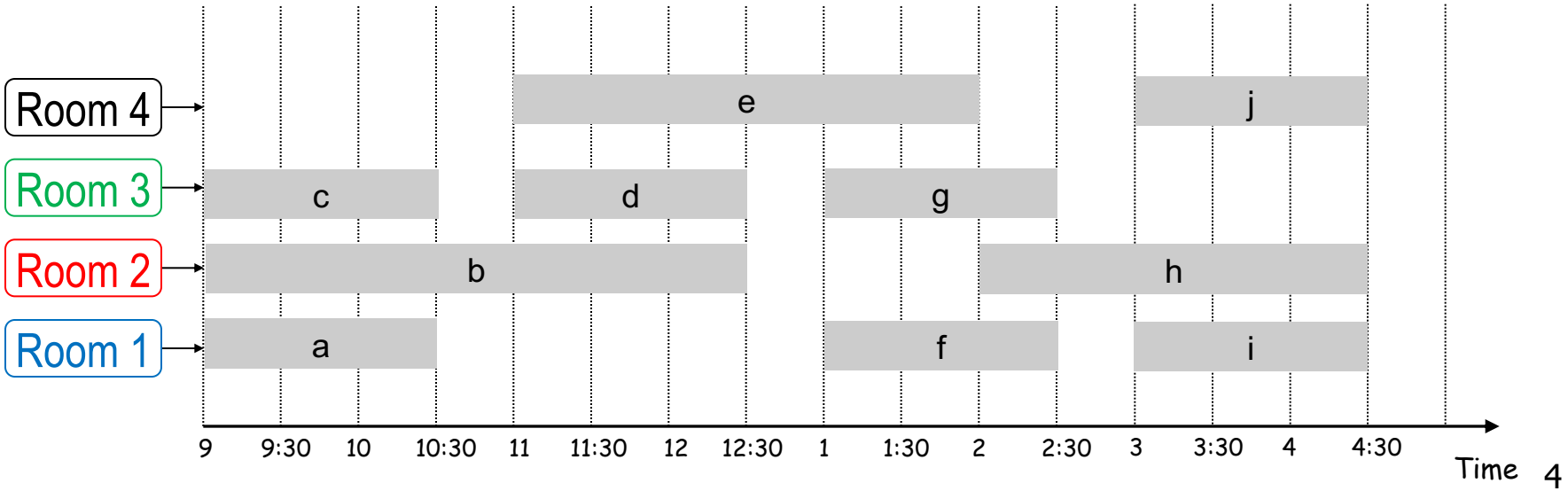
- Recipe:
  - Order the input in some way (the most ‘important’ element will be considered first)
  - Go through the input according to the order
  - Determine the strategy to construct best current partial solution in each step

Interval Partitioning Technique: Structural

Interval Partitioning

Lecture j starts at $s(j)$ and finishes at $f(j)$.

Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.



A Greedy Algorithm

Greedy algorithm: Consider lectures in increasing order of start time: assign lecture to any compatible classroom.

```
Sort intervals by starting time so that s1 ≤ s2 ≤ ... ≤ sn.
d ← 0

for j = 1 to n {
   if (lect j is compatible with some classroom k, 1 ≤ k ≤ d)
      schedule lecture j in classroom k
   else
      allocate a new classroom d + 1
      schedule lecture j in classroom d + 1
      d ← d + 1
}
```

Implementation: Exercise!

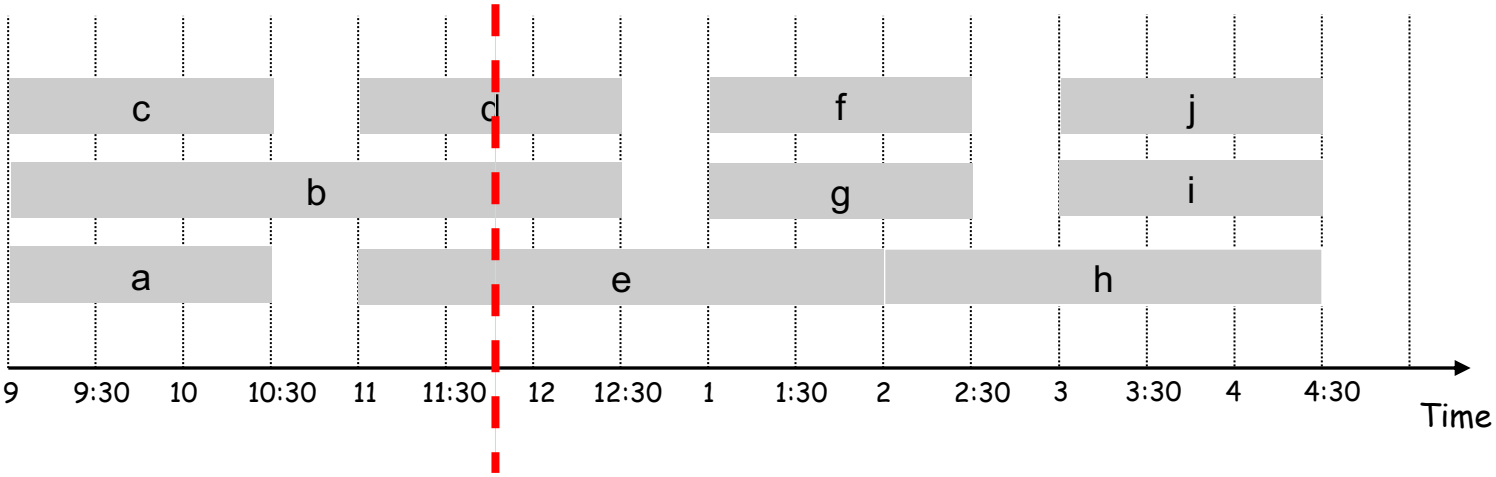
A Structural Lower-Bound on OPT

Def. The depth of a set of intervals is the maximum number that contain any given time.

Key observation. Number of classrooms needed $\geq$ depth.

Ex: Depth of schedule below $= 3 \Rightarrow$ schedule below is optimal.

Q. Does there always exist a schedule equal to depth of intervals?



Correctness

Observation: Greedy algorithm never schedules two incompatible lectures in the same classroom.

Theorem: Greedy algorithm is optimal.

Proof (exploit structural property).

Let d = number of classrooms that the greedy algorithm allocates.

Classroom d is opened because we needed to schedule a job, say j, that is incompatible with all $d - 1$ previously used classrooms.

Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than $s(j)$.

Thus, we have d lectures overlapping at time $s(j) + \epsilon$, i.e. depth $\geq d$

"OPT Observation" $\Rightarrow$ $d \geq$ depth,

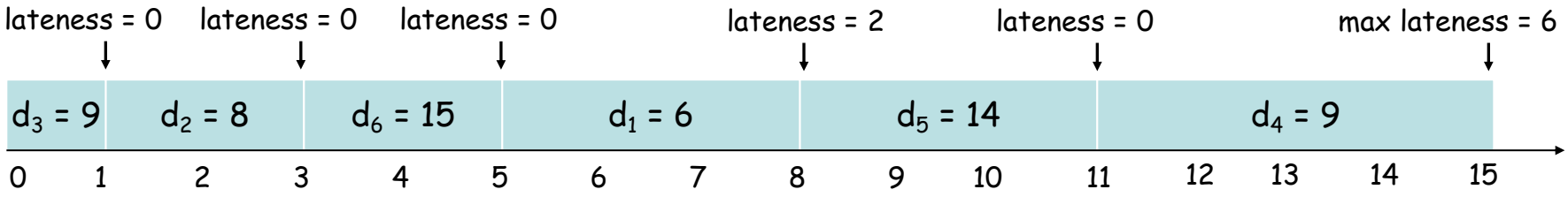
so d = depth and greedy is optimal

Lateness Minimization
Technique: Exchange Argument

Scheduling to Minimizing Lateness

- Instead of start and finish times, job i has
  - Time Requirement t_i which must be scheduled in a contiguous block
  - Deadline d_i by which time the job would like to be finished
- Jobs are scheduled into time intervals $[s_i, f_i]$ s.t. $t_i = f_i - s_i$.
- Lateness for job i is
  - If $d_i < f_i$ then job i is late by $L_i = f_i - d_i$ otherwise its lateness $L_i = 0$
- **Goal:** Find a schedule that minimize the Maximum lateness $L = \max_i L_i$

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



Minimizing Lateness: Greedy Algorithms

Greedy template. Consider jobs in some order.

- [Shortest processing time first]
  Consider jobs in ascending order of processing time t_j.

- [Smallest slack]
  Consider jobs in ascending order of slack $d_j - t_j$.

	1	2
t_j	1	10
d_j	100	10

counterexample

- [Earliest deadline first]
  Consider jobs in ascending order of deadline d_j.

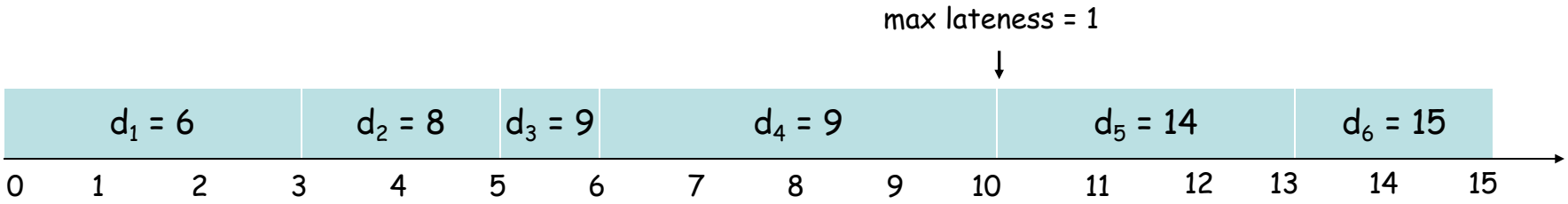
	1	2
t_j	1	10
d_j	2	10

counterexample

Greedy Algorithm: Earliest Deadline First

Sort deadlines in increasing order $(d_1 \le d_2 \le \cdots \le d_n)$
$f \leftarrow 0$
for $i \leftarrow 1$ to n to
    $s_i \leftarrow f$
    $f_i \leftarrow s_i + t_i$
    $f \leftarrow f_i$
end for

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



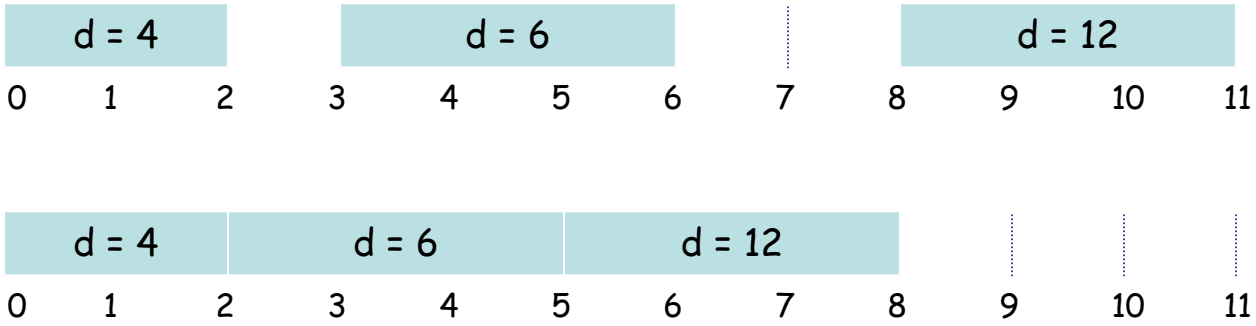
Proof for Greedy Algorithm: Exchange Argument

- We will show that if there is another schedule $\boldsymbol{O}$ (think optimal schedule) then we can gradually change $\boldsymbol{O}$ so that
  - at each step the maximum lateness in $\boldsymbol{O}$ never gets worse
  - it eventually becomes the same cost as $\boldsymbol{A}$

Minimizing Lateness: No Idle Time

Observation.

- There exists an optimal schedule with no idle time.



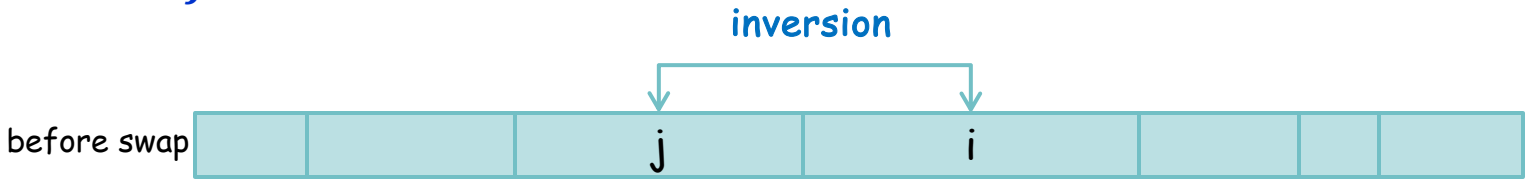
Observation.

- The greedy schedule has no idle time.

Minimizing Lateness: Inversions

Definition

- An inversion in schedule S is a pair of jobs i and j such that $d_i < d_j$ but j scheduled before i.



Observation

- Greedy schedule has no inversions.

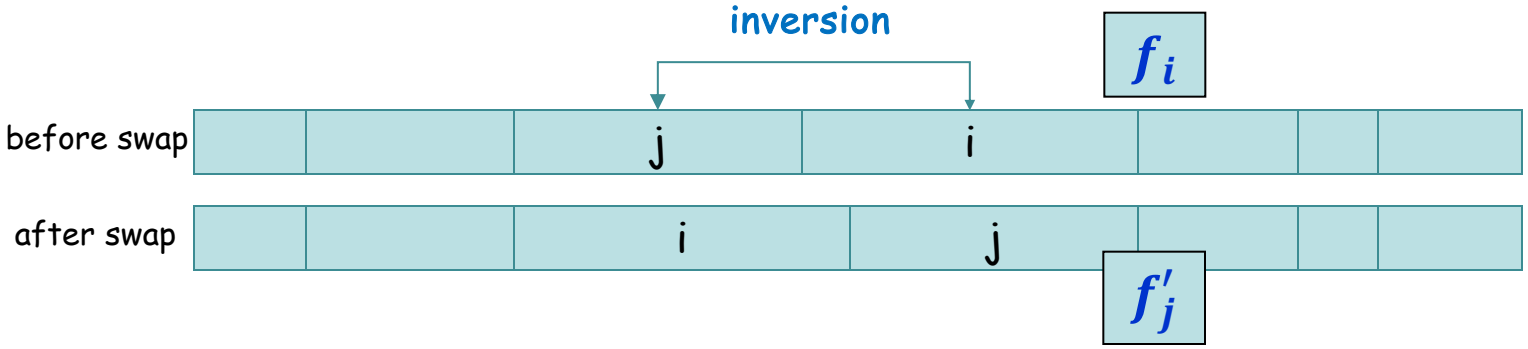
Observation

- If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled **consecutively**.
  - Why? If no inversion, then $d_i \leq d_{i+1}$ for all i .

Minimizing Lateness: Inversions

Definition

- An inversion in schedule S is a pair of jobs i and j such that $d_i < d_j$ but j scheduled before i.

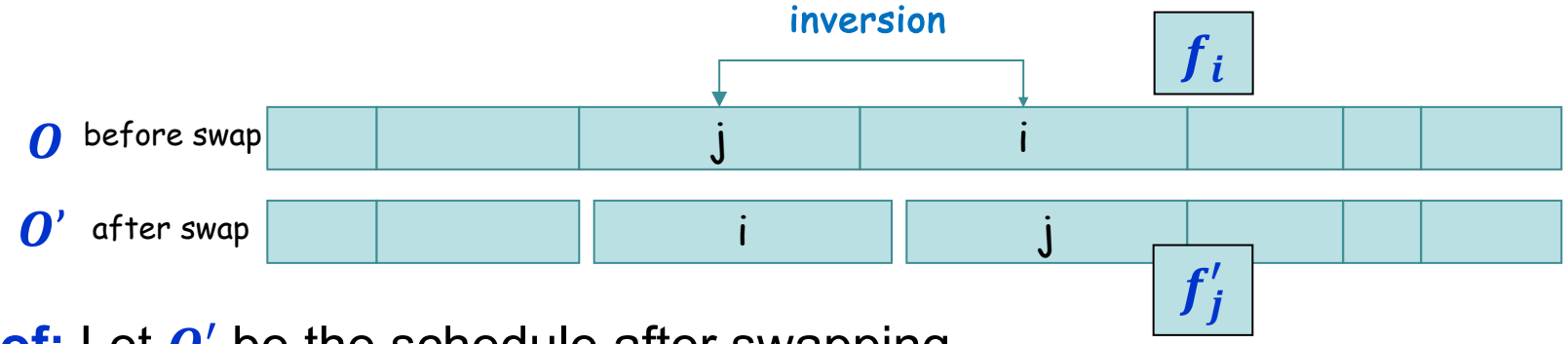


Claim

- Swapping two adjacent, inverted jobs reduces the number of inversions by one and does not increase the max lateness.

Minimizing Lateness: Inversions

Lemma: Swapping two adjacent, inverted jobs does not increase the maximum lateness.



Proof: Let O' be the schedule after swapping.

- All other jobs $k \neq i, j$ have $L_k' = L_k$
- Lateness $L_i' \leq L_i$ since i is scheduled earlier in O' than in O
- Jobs i and j together occupy the same total time slot in both schedules
  - $f_j' = f_i$ so $L_j' = f_j' - d_j = f_i - d_j < f_i - d_i = L_i$
- Maximum lateness has not increased!

Optimal schedules and inversions

Did we finish the proof for greedy?

Claim: There is an optimal schedule with no idle time and no inversions

Proof:

- By previous argument there is an optimal schedule $\boldsymbol{O}$ with no idle time
- If $\boldsymbol{O}$ has an inversion then it has a **consecutive** pair of jobs in its schedule that are inverted and can be swapped without increasing lateness
- Eventually these swaps will produce an optimal schedule with no inversions
  - Each swap decreases the number of inversions by **1**
  - There are at most $\boldsymbol{n(n-1)/2}$ inversions. (we only care that this is finite.)

Idleness and Inversions are the only issue

Claim: All schedules with no inversions and no idle time have the same maximum lateness

Proof:

- Schedules can differ only in how they order jobs with equal deadlines
- Consider all jobs having some common deadline d
- Maximum lateness of these jobs is based only on the finish time of the last of these jobs but the set of these jobs occupies the same time segment in both schedules
  - Last of these jobs finishes at the same time in any such schedule.

Earliest Deadline First is optimal

We know that

- There is an optimal schedule with no idle time or inversions
- All schedules with no idle time or inversions have the same maximum lateness
- EDF produces a schedule with no idle time or inversions

Therefore

- EDF produces an optimal schedule

Life Wisdom:

- Finish your jobs according to deadline!
- ☹ Unfortunately, we don't see all jobs when born.