CS 401: Computer Algorithm I

Lateness Minimization / Single Source Shortest Path

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Stuff

- Homework 2 submission deadline extends to February 27

- In class midterm exam: February 29 Thursday 12:30pm – 1:45pm

- Midterm review and details of the exam: February 27

Recap and Outline

- Greedy algorithm:
  - Divide solution construction into small steps
  - Decision in each step: ‘best’ current partial solution at each step

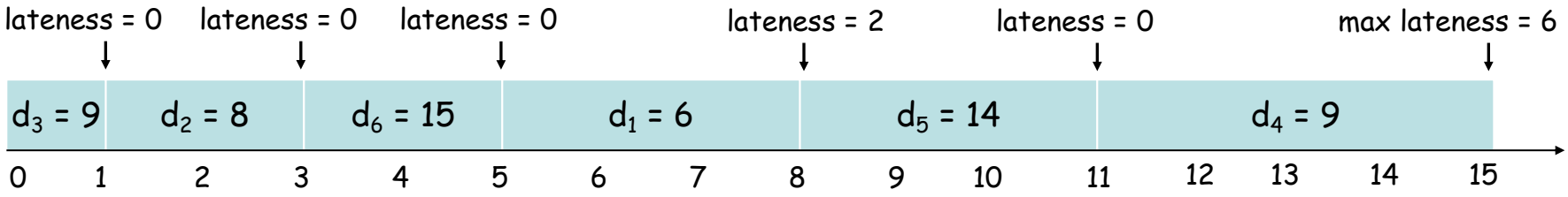
- Recipe:
  - Order the input in some way (the most ‘important’ element will be considered first)
  - Go through the input according to the order
  - Determine the strategy to construct best current partial solution in each step

Lateness Minimization
Technique: Exchange Argument

Scheduling to Minimizing Lateness

- Instead of start and finish times, job i has
  - Time Requirement t_i which must be scheduled in a contiguous block
  - Deadline d_i by which time the job would like to be finished
- Jobs are scheduled into time intervals $[s_i, f_i]$ s.t. $t_i = f_i - s_i$.
- Lateness for job i is
  - If $d_i < f_i$ then job i is late by $L_i = f_i - d_i$ otherwise its lateness $L_i = 0$
- **Goal:** Find a schedule that minimize the Maximum lateness $L = \max_i L_i$

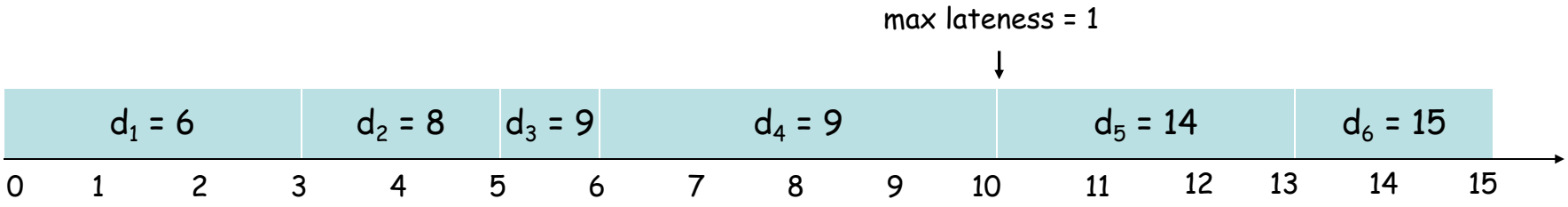
	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



Greedy Algorithm: Earliest Deadline First

Sort deadlines in increasing order $(d_1 \leq d_2 \leq \cdots \leq d_n)$
$f \leftarrow 0$
for $i \leftarrow 1$ to n to
    $s_i \leftarrow f$
    $f_i \leftarrow s_i + t_i$
    $f \leftarrow f_i$
end for

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15



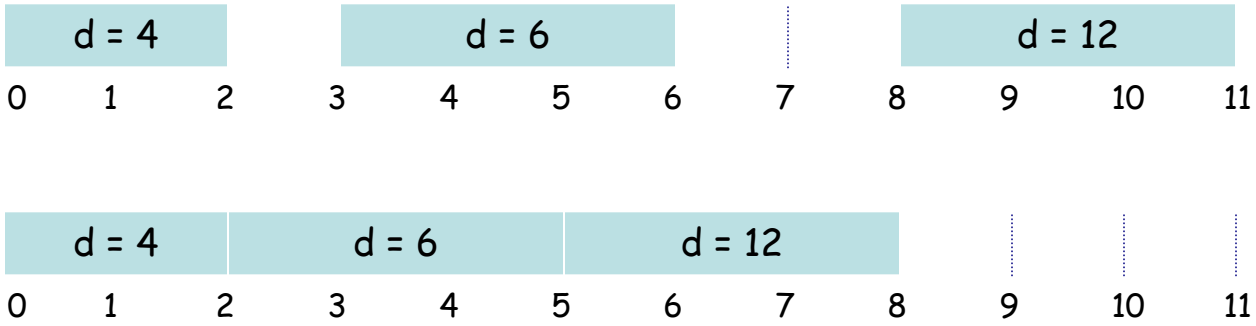
Proof for Greedy Algorithm: Exchange Argument

- We will show that if there is another schedule $\boldsymbol{O}$ (think optimal schedule) then we can gradually change $\boldsymbol{O}$ so that
  - at each step the maximum lateness in $\boldsymbol{O}$ never gets worse
  - it eventually becomes the same cost as $\boldsymbol{A}$

Minimizing Lateness: No Idle Time

Observation.

- There exists an optimal schedule with no idle time.



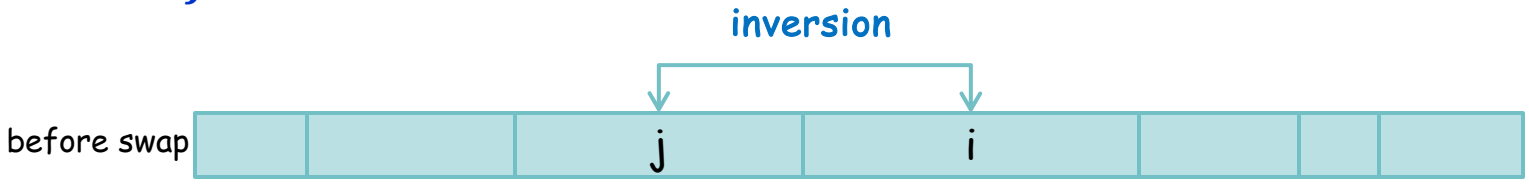
Observation.

- The greedy schedule has no idle time.

Minimizing Lateness: Inversions

Definition

- An inversion in schedule S is a pair of jobs i and j such that $d_i < d_j$ but j scheduled before i.



Observation

- Greedy schedule has no inversions.

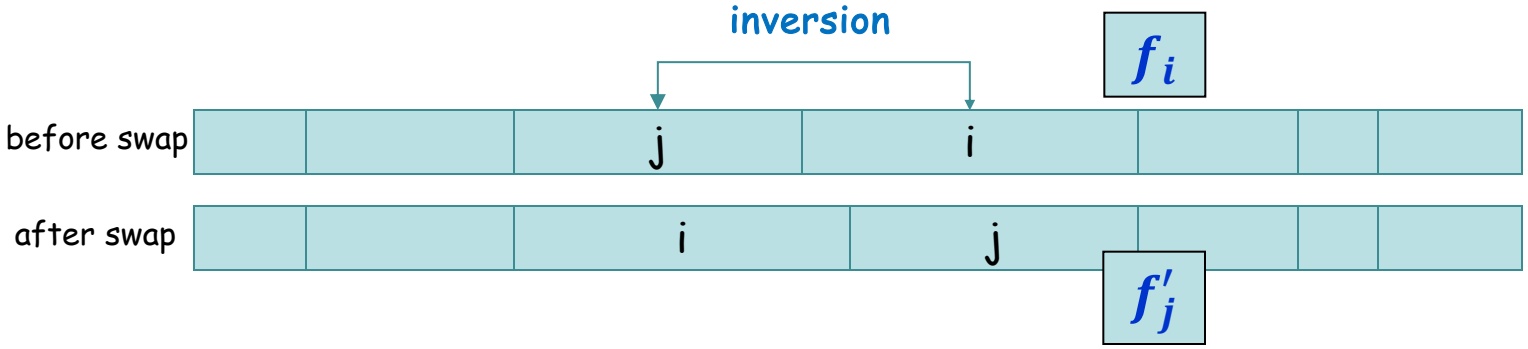
Observation

- If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled **consecutively**.
  - Why? If no inversion, then $d_i \leq d_{i+1}$ for all i .

Minimizing Lateness: Inversions

Definition

- An inversion in schedule S is a pair of jobs i and j such that $d_i < d_j$ but j scheduled before i.

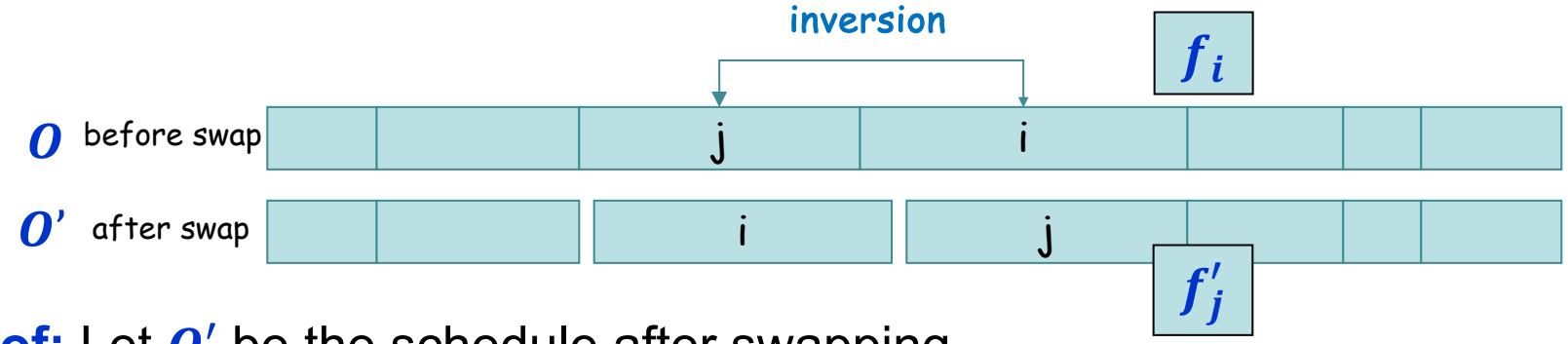


Claim

- Swapping two adjacent, inverted jobs reduces the number of inversions by one and does not increase the max lateness.

Minimizing Lateness: Inversions

Lemma: Swapping two adjacent, inverted jobs does not increase the maximum lateness.



Proof: Let O' be the schedule after swapping.

- All other jobs $k \neq i, j$ have $L_k' = L_k$
- Lateness $L_i' \leq L_i$ since i is scheduled earlier in O' than in O
- Jobs i and j together occupy the same total time slot in both schedules
  - $f_j' = f_i$ so $L_j' = f_j' - d_j = f_i - d_j < f_i - d_i = L_i$
- Maximum lateness has not increased!

Optimal schedules and inversions

Did we finish the proof for greedy?

Claim: There is an optimal schedule with no idle time and no inversions

Proof:

- By previous argument there is an optimal schedule $\boldsymbol{O}$ with no idle time
- If $\boldsymbol{O}$ has an inversion then it has a **consecutive** pair of jobs in its schedule that are inverted and can be swapped without increasing lateness
- Eventually these swaps will produce an optimal schedule with no inversions
  - Each swap decreases the number of inversions by **1**
  - There are at most $\boldsymbol{n(n-1)/2}$ inversions. (we only care that this is finite.)

Idleness and Inversions are the only issue

Claim: All schedules with no inversions and no idle time have the same maximum lateness

Proof:

- Schedules can differ only in how they order jobs with equal deadlines
- Consider all jobs having some common deadline d
- Maximum lateness of these jobs is based only on the finish time of the last of these jobs but the set of these jobs occupies the same time segment in both schedules
  - Last of these jobs finishes at the same time in any such schedule.

Earliest Deadline First is optimal

We know that

- There is an optimal schedule with no idle time or inversions
- All schedules with no idle time or inversions have the same maximum lateness
- EDF produces a schedule with no idle time or inversions

Therefore

- EDF produces an optimal schedule

Life Wisdom:

- Finish your jobs according to deadline!
- ☹ Unfortunately, we don't see all jobs when born.

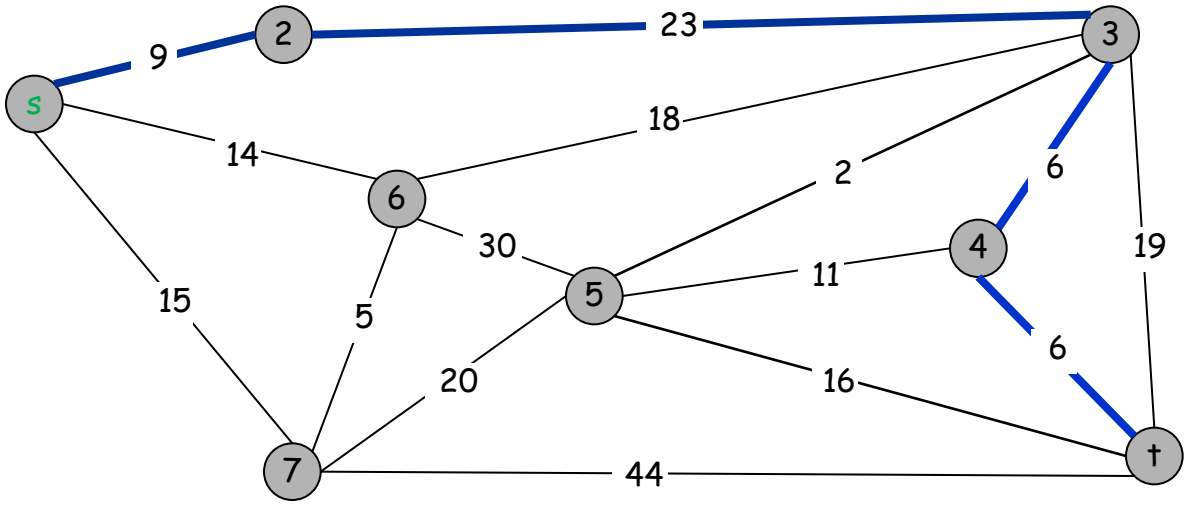
Single Source Shortest Path

Single Source Shortest Path

Given an (un)directed connected graph $G = (V, E)$ with non-negative edge weights $c_e \geq 0$ and a start vertex s.

Find length of shortest paths from s to each vertex in G

length of path = sum of edge weights in path



Cost of path s-2-3-4-t
= 9 + 23 + 6 + 6
= 44.

Single Source Shortest Path

Greedy Recipe:

- Order the input in some way (the most ‘important’ element will be considered first)
- Go through the input according to the order
- Determine the strategy to construct best current partial solution in each step

For shortest path: How do we order the input?

- Idea: Instead of ordering the input, we order the output

Dijkstra's Algorithm

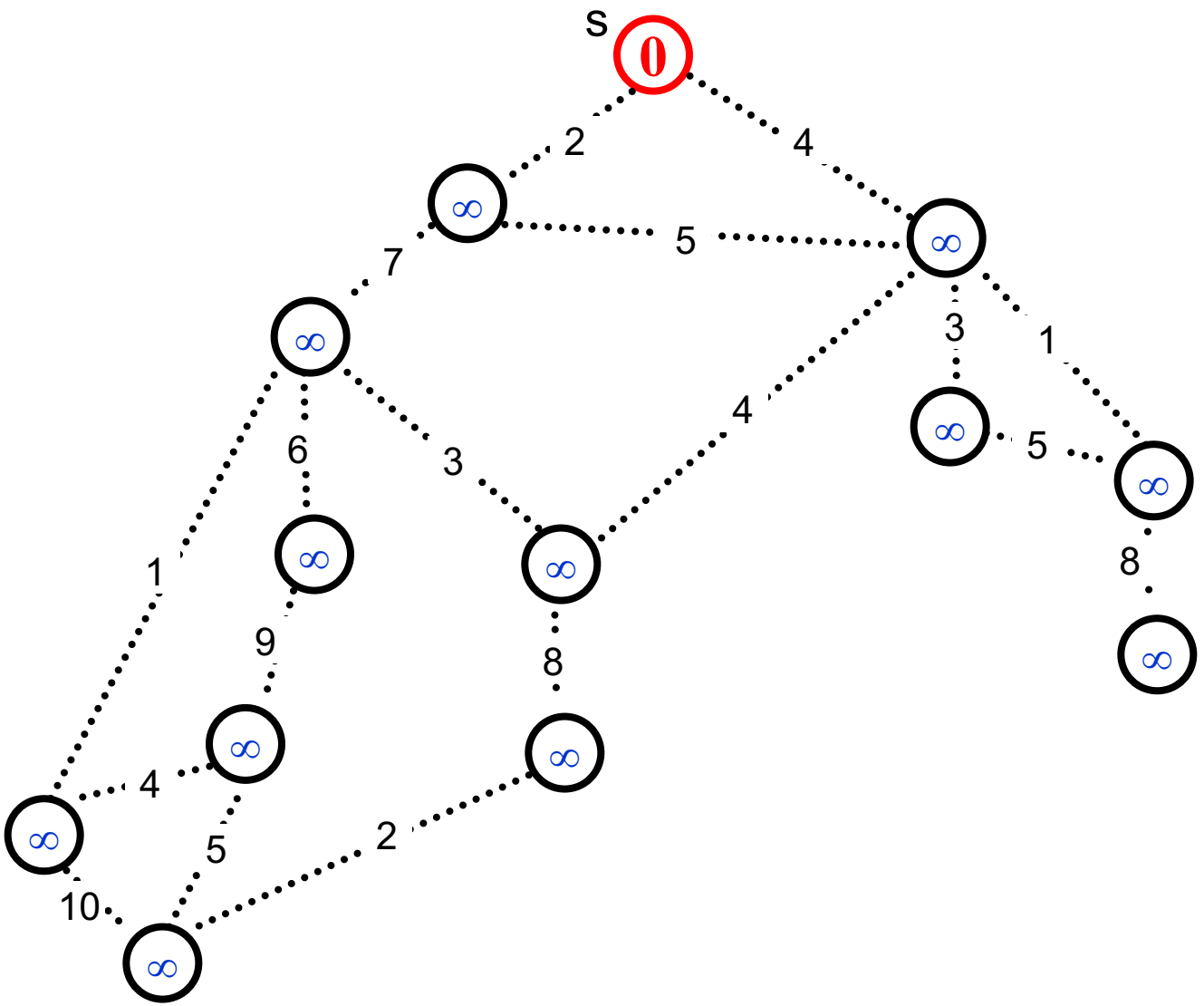
```
Dijkstra(G, c, s) {
    Initialize set of explored nodes S ← {s}

    // Maintain distance from s to each vertices in S
    d[s] ← 0

    while (S ≠ V)
    {
        Pick an edge (u, v) such that u ∈ S and v ∉ S and
                d[u] + c_(u,v) is as small as possible.

        Add v to S and define d[v] = d[u] + c_(u,v).
        Parent(v) ← u.
    }
```

Dijkstra's Algorithm: Example



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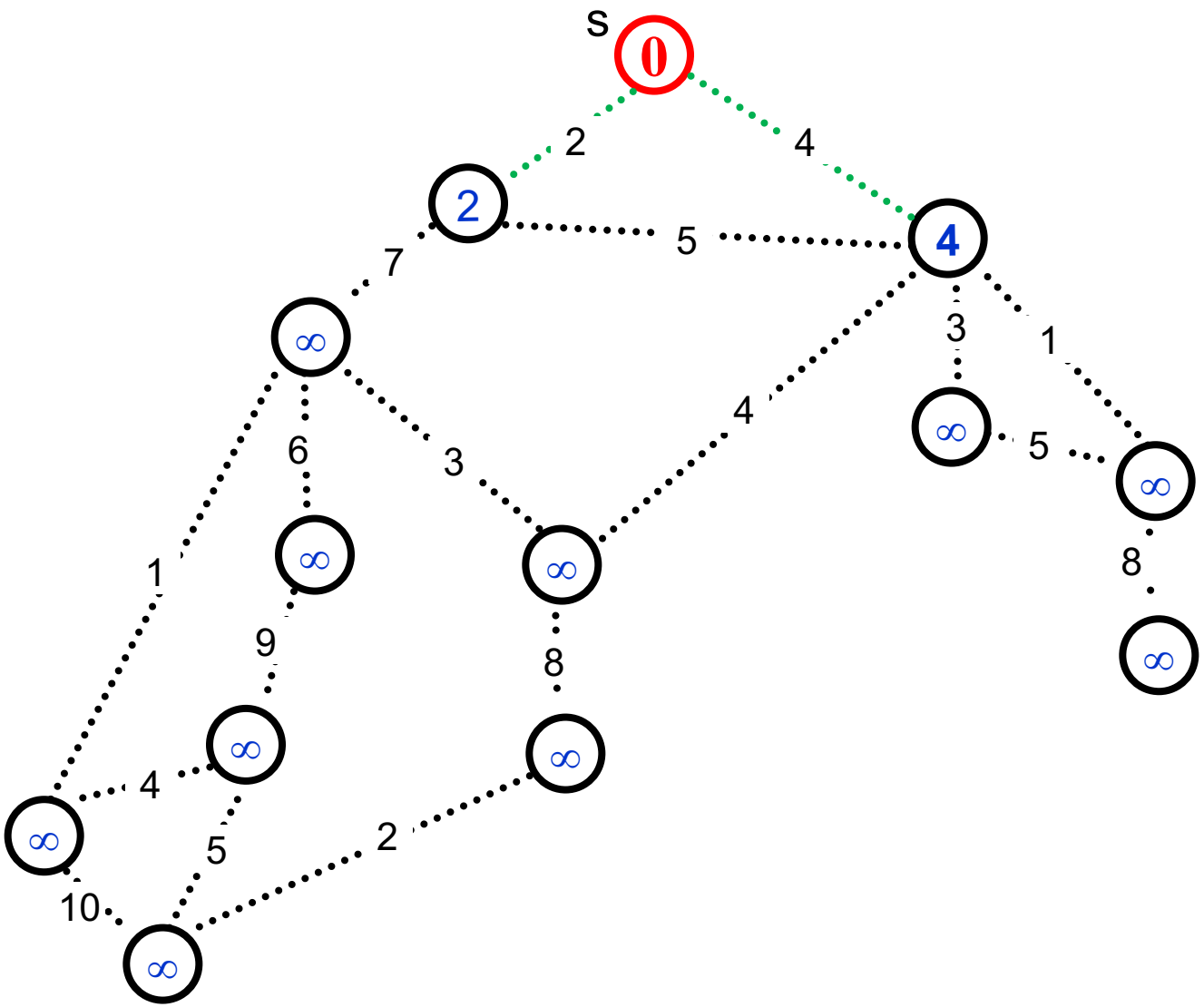

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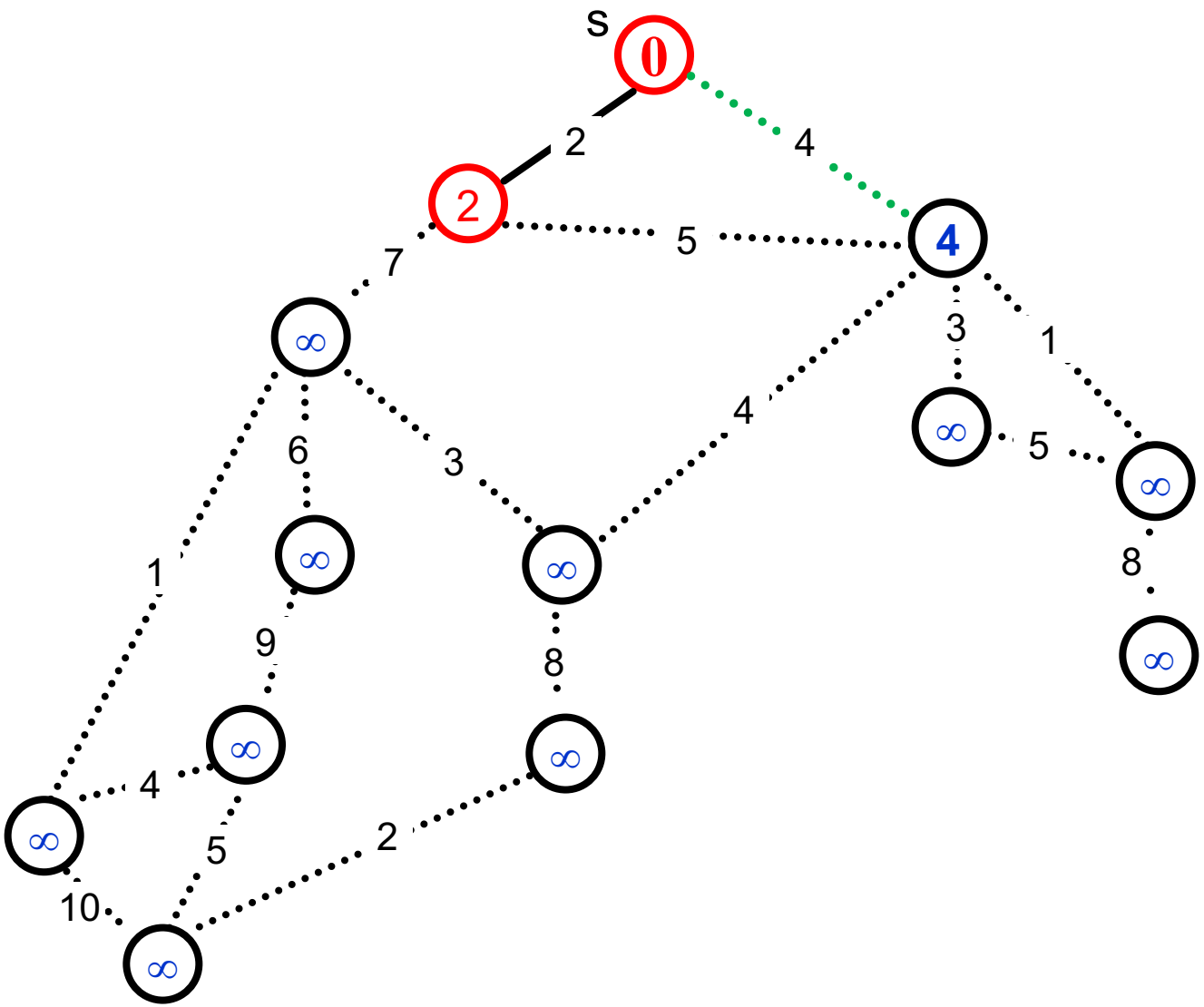

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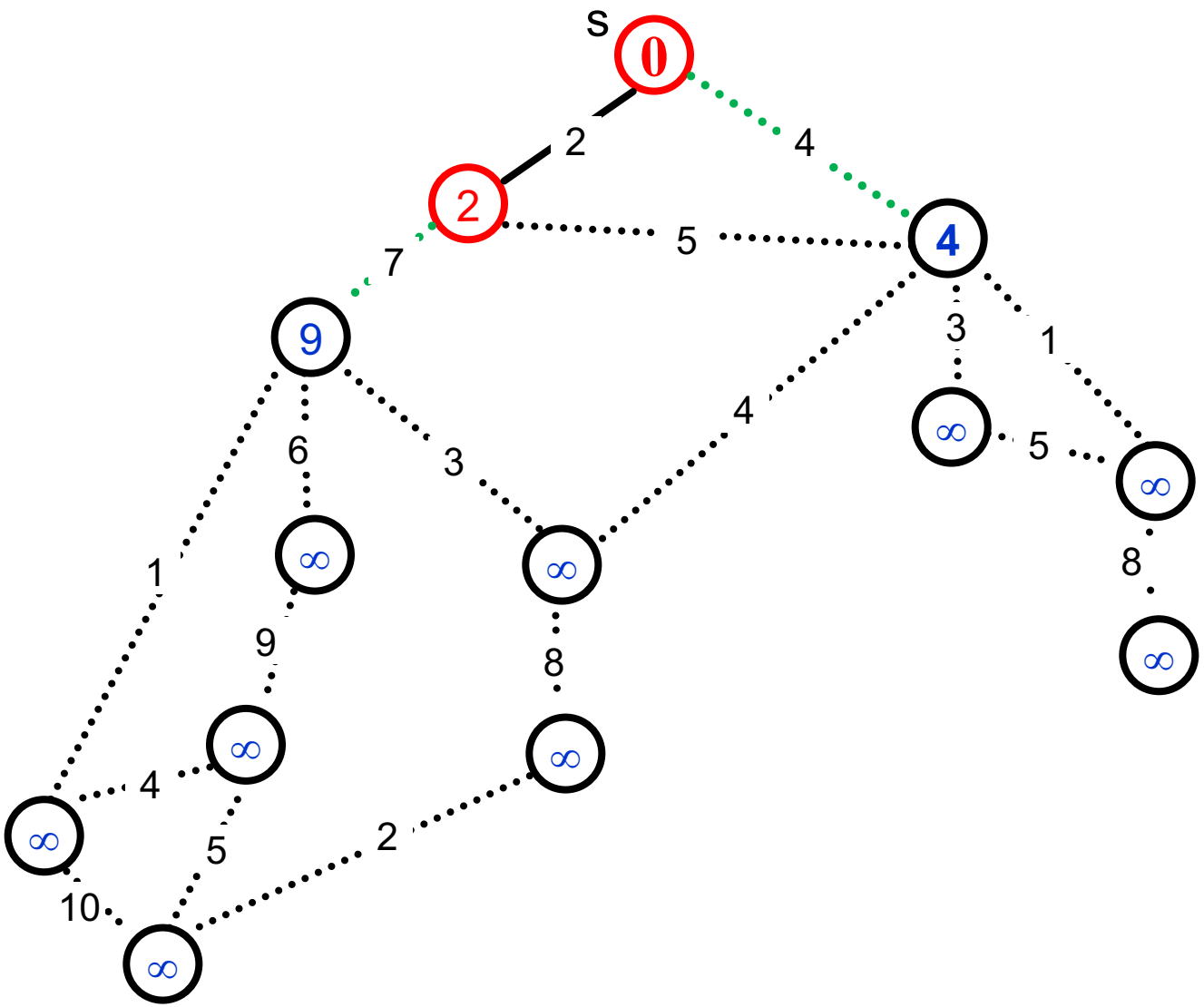

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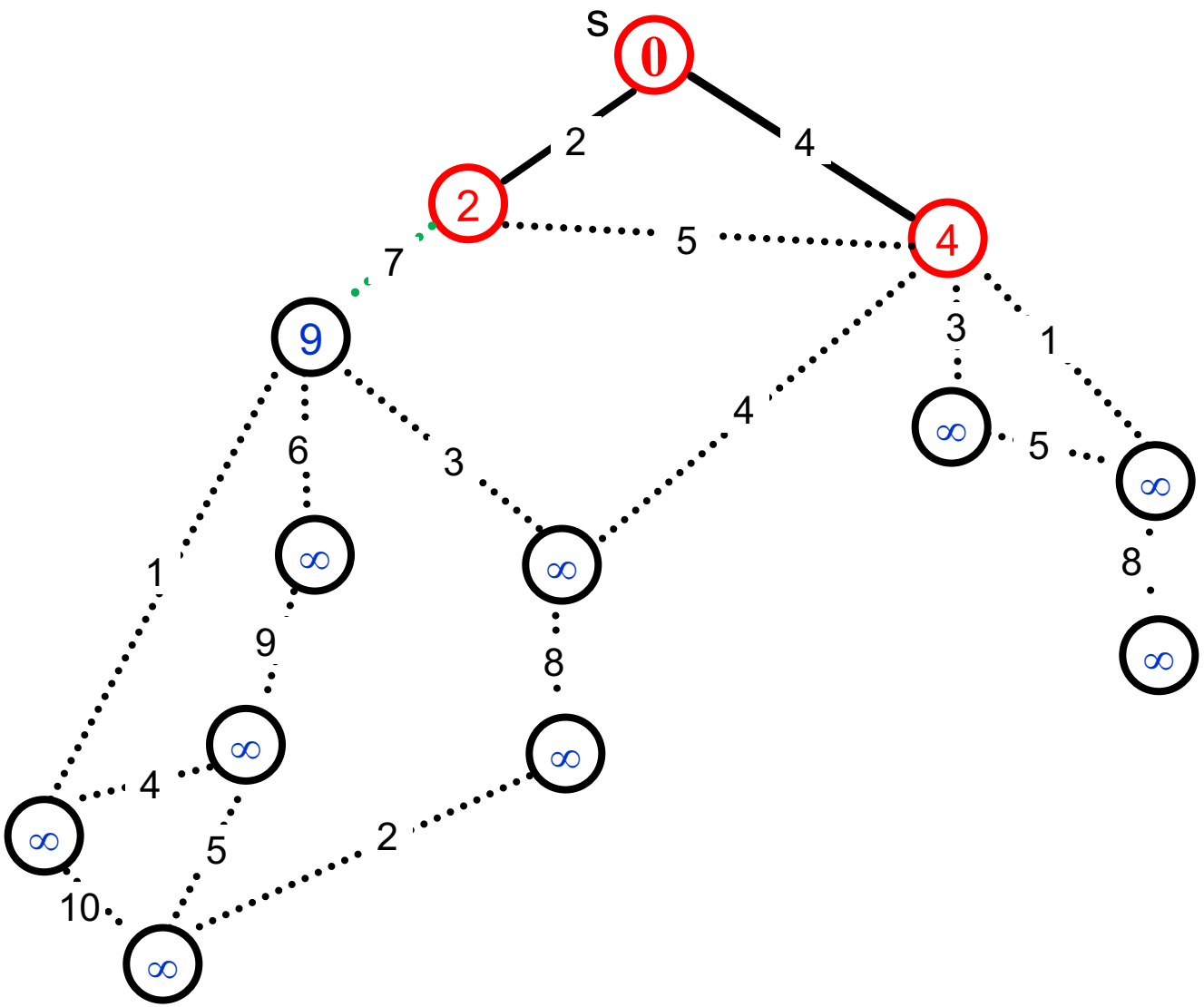

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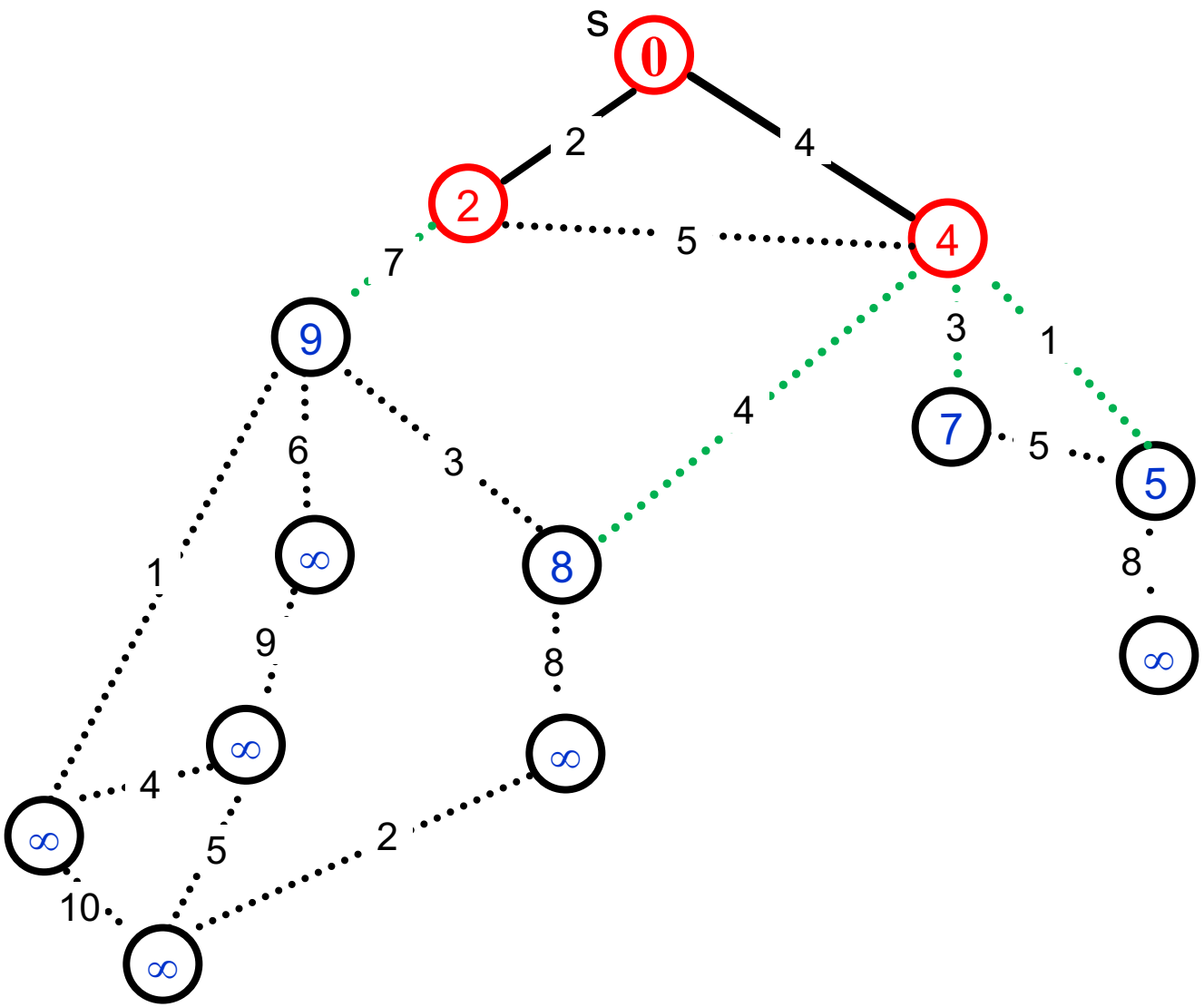

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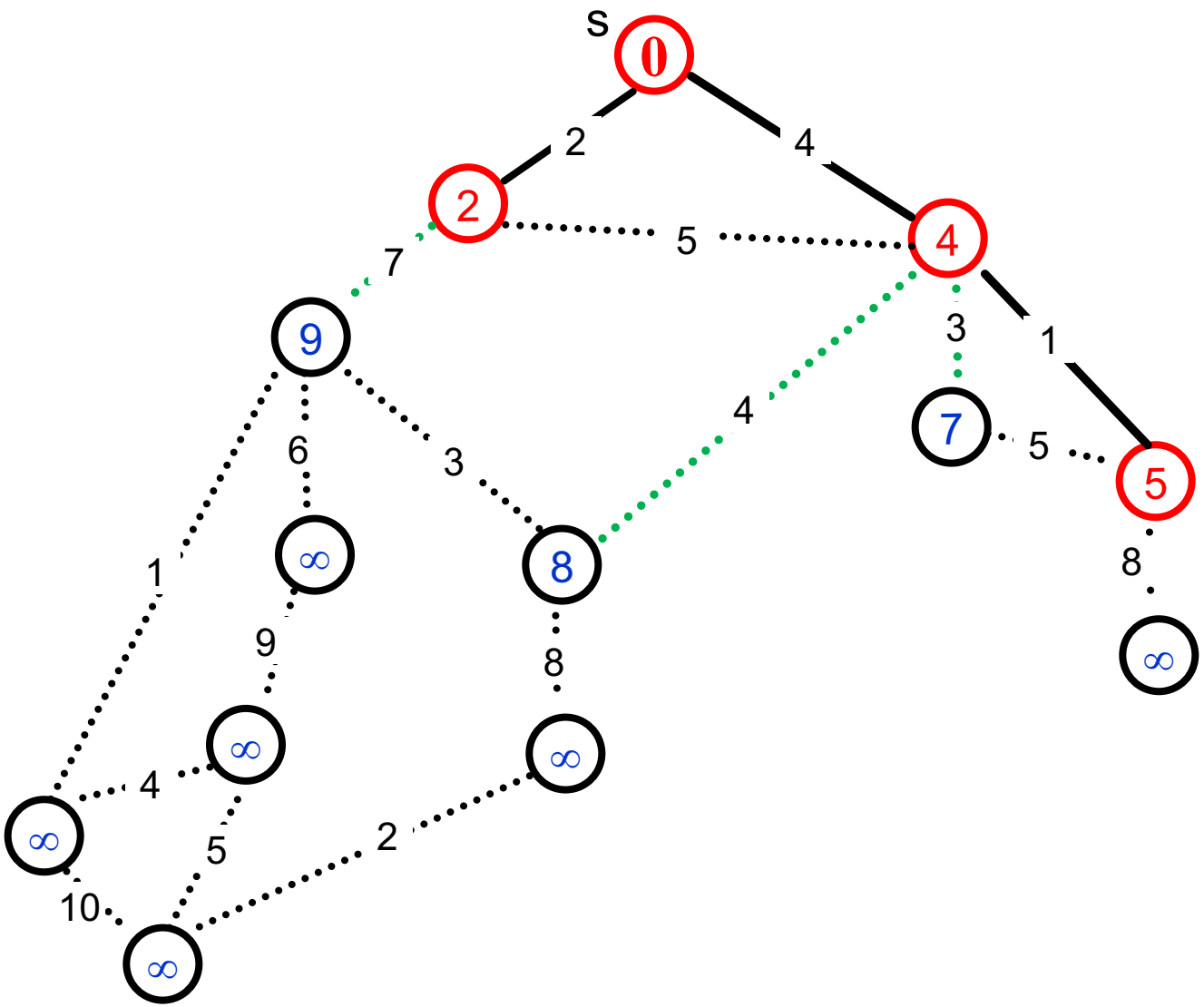

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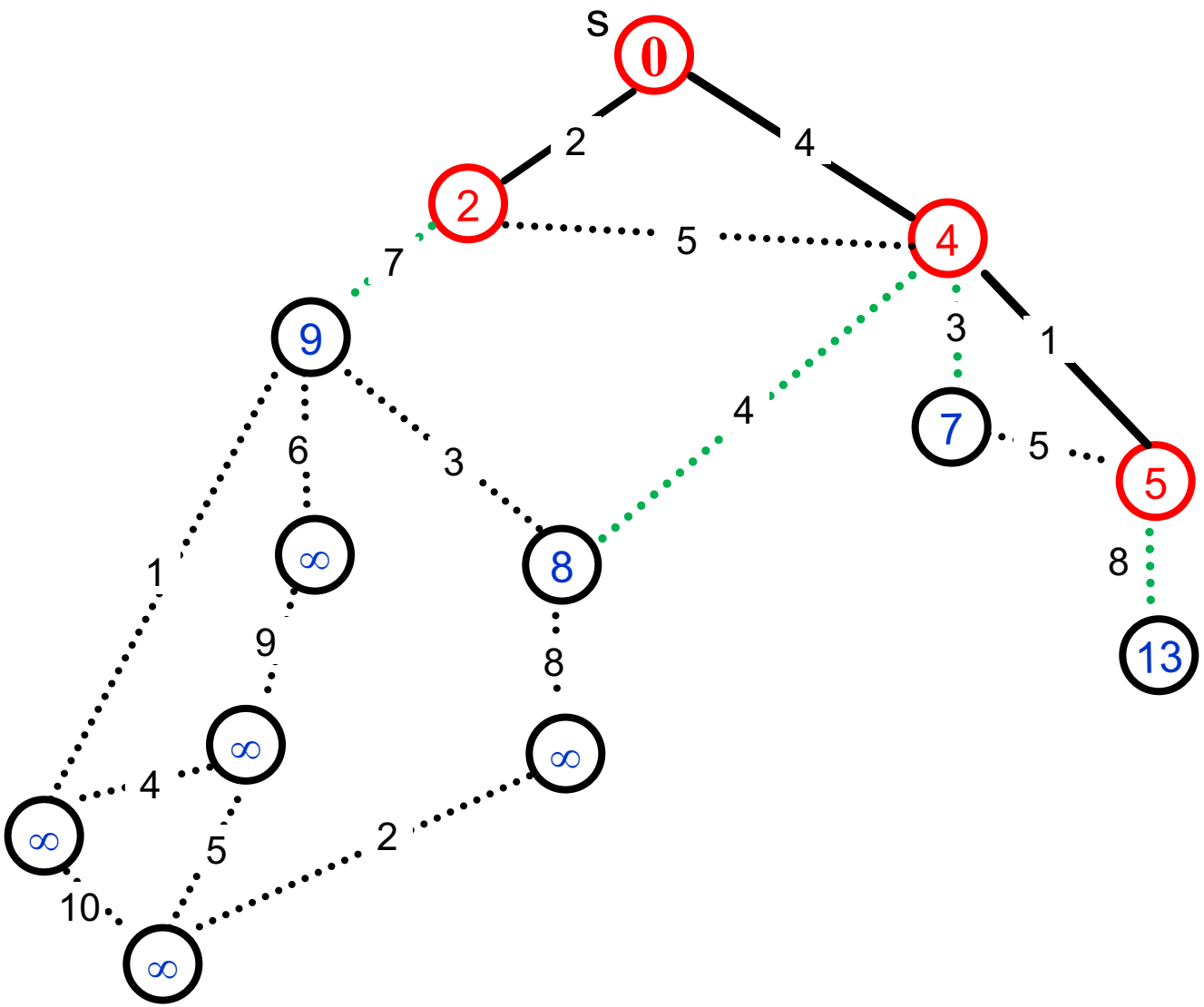

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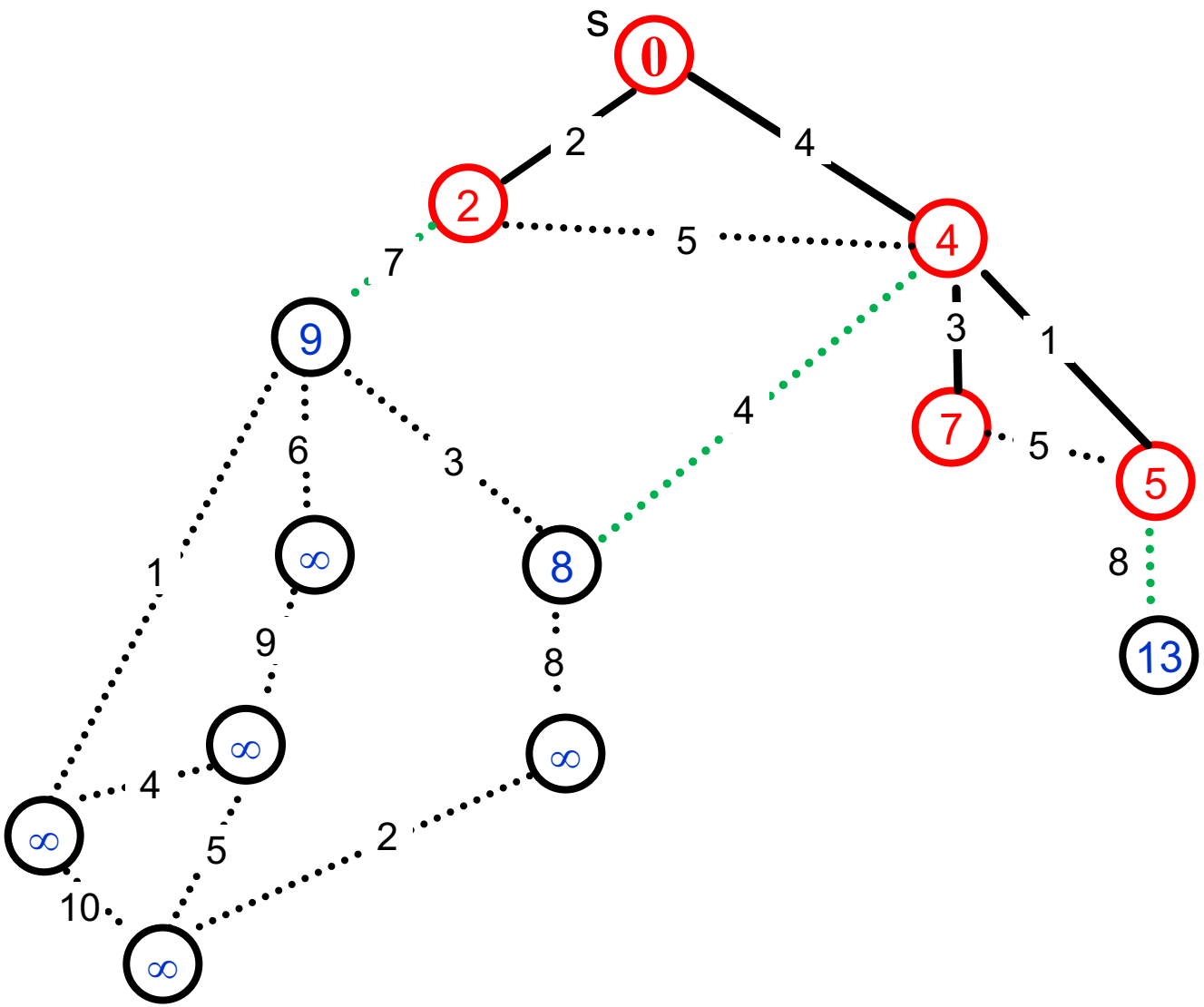

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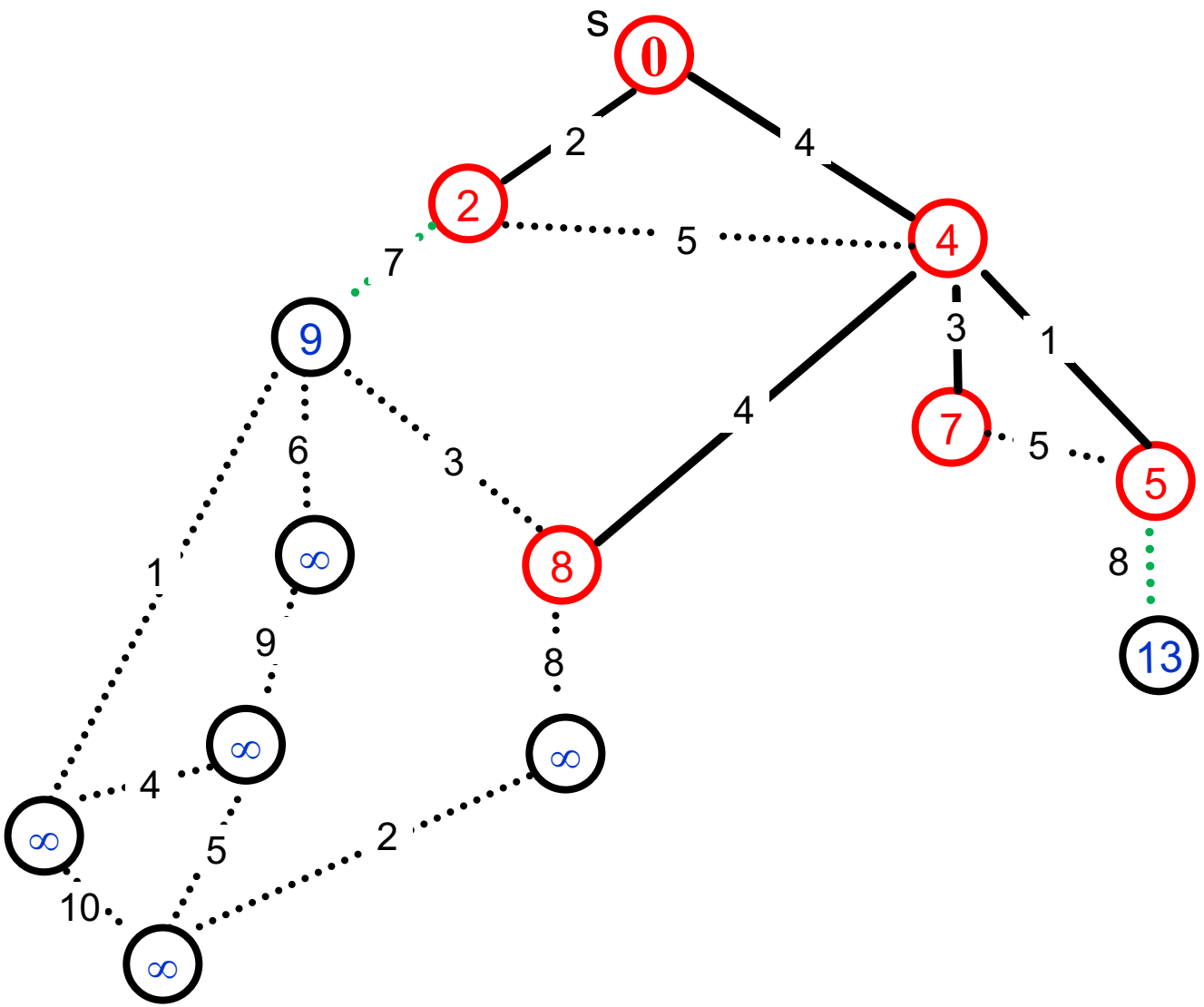

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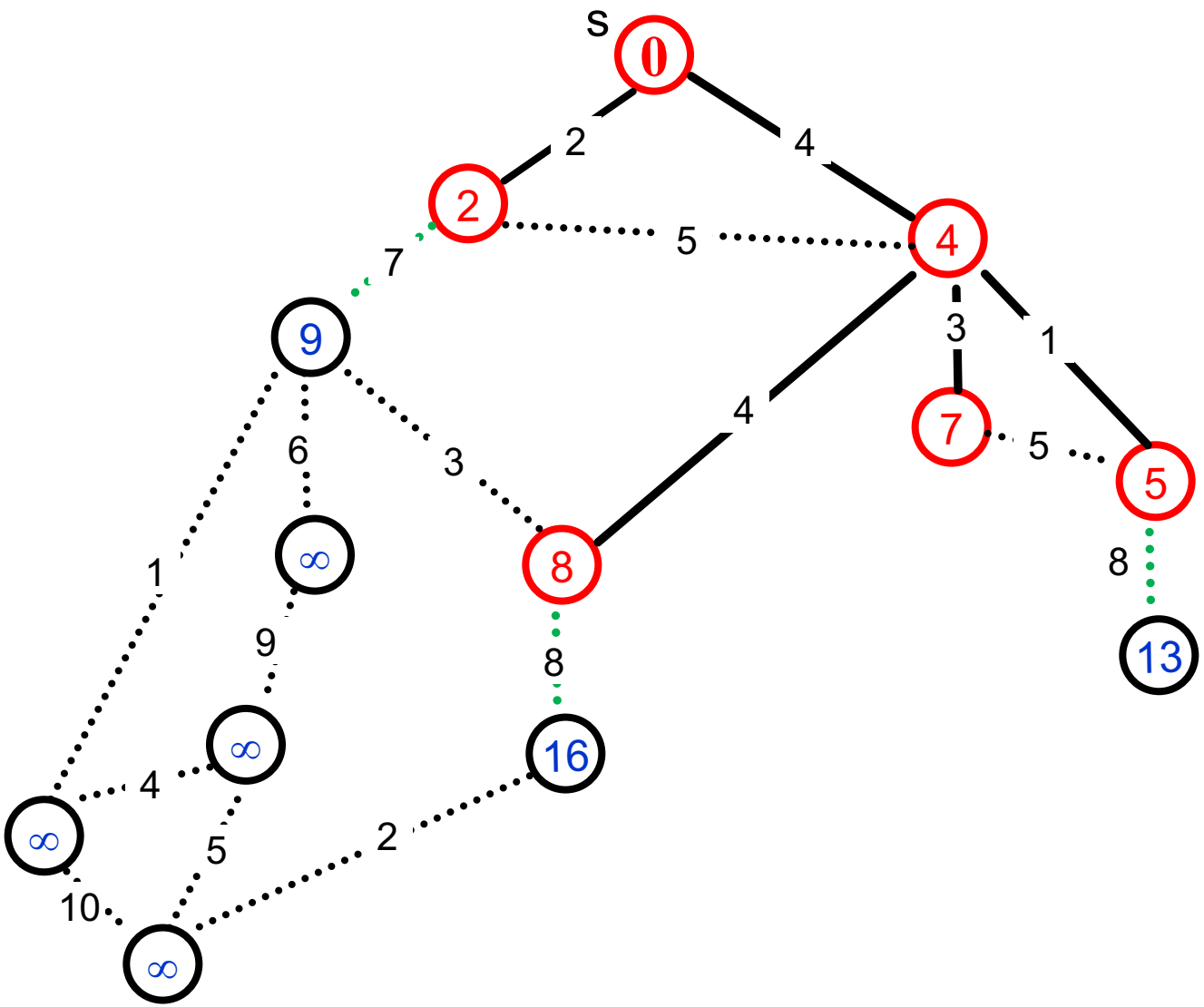

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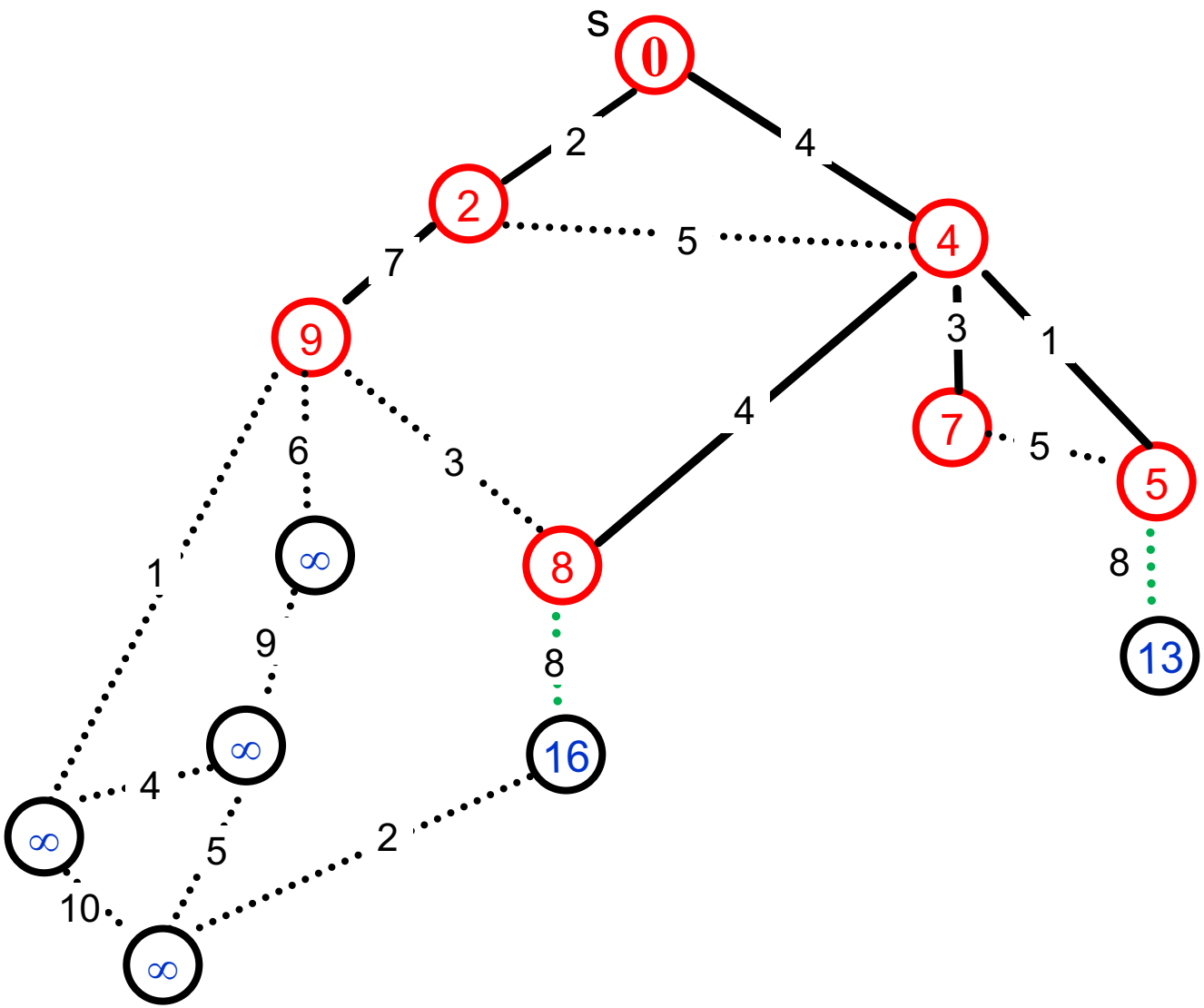

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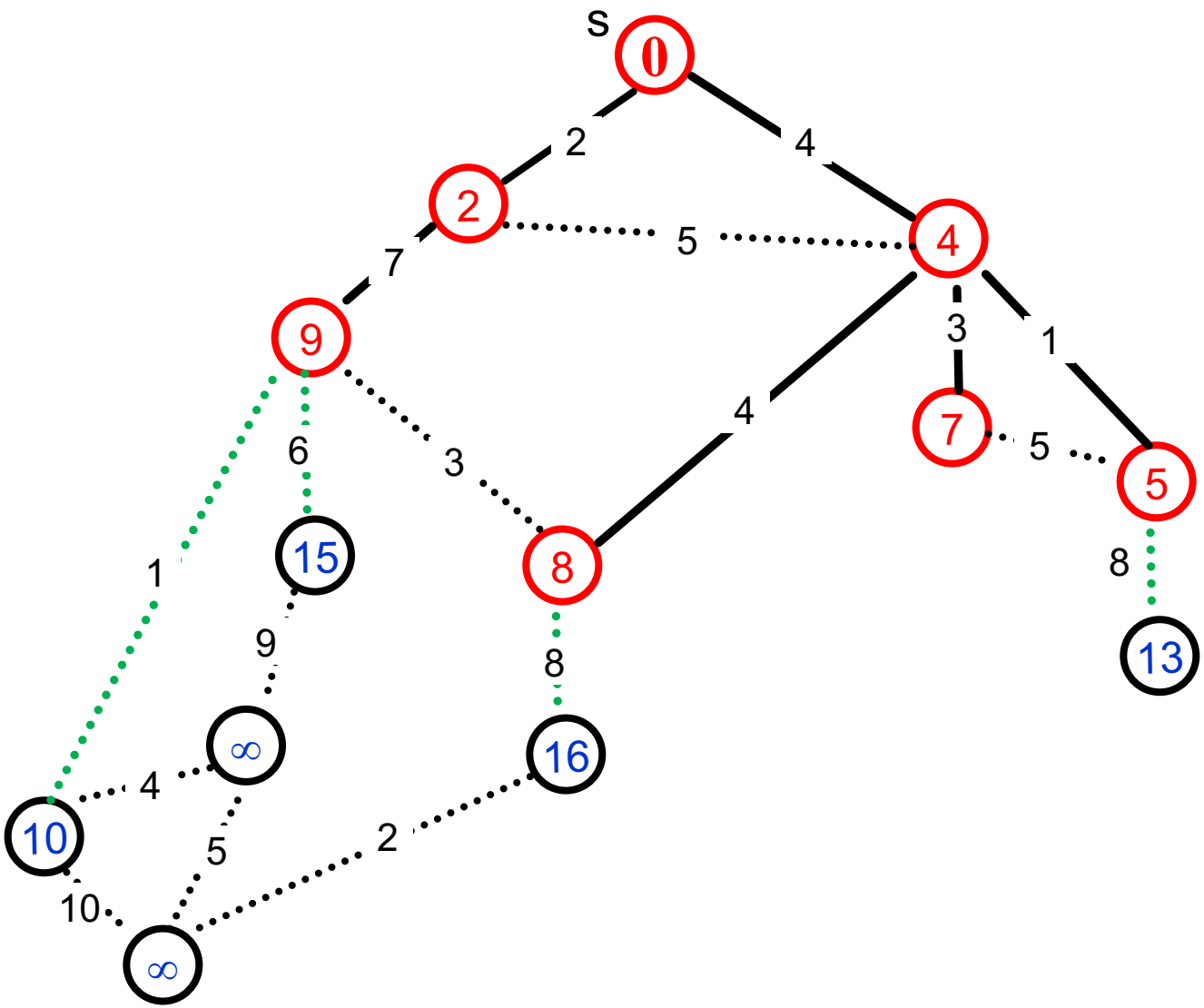

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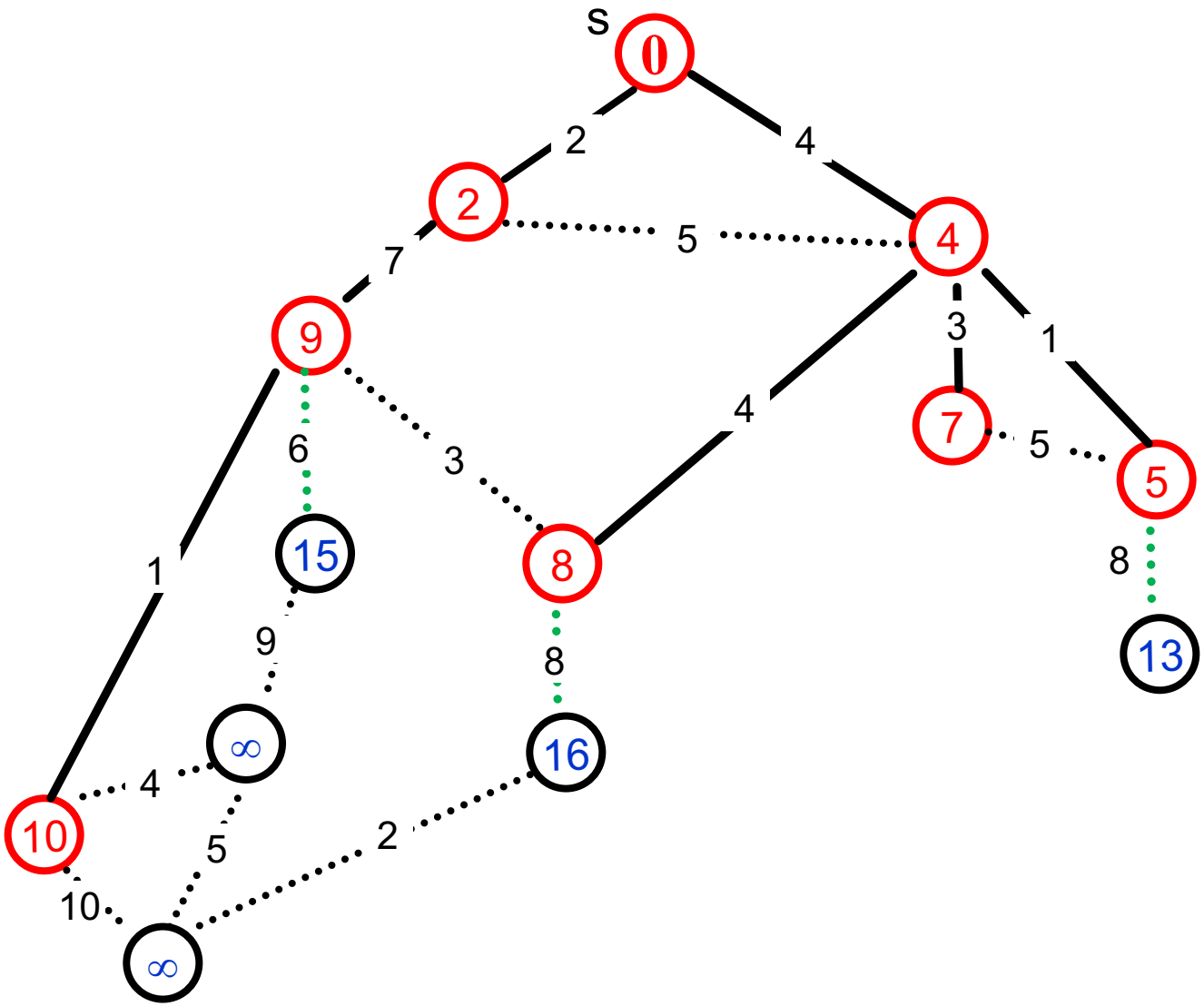

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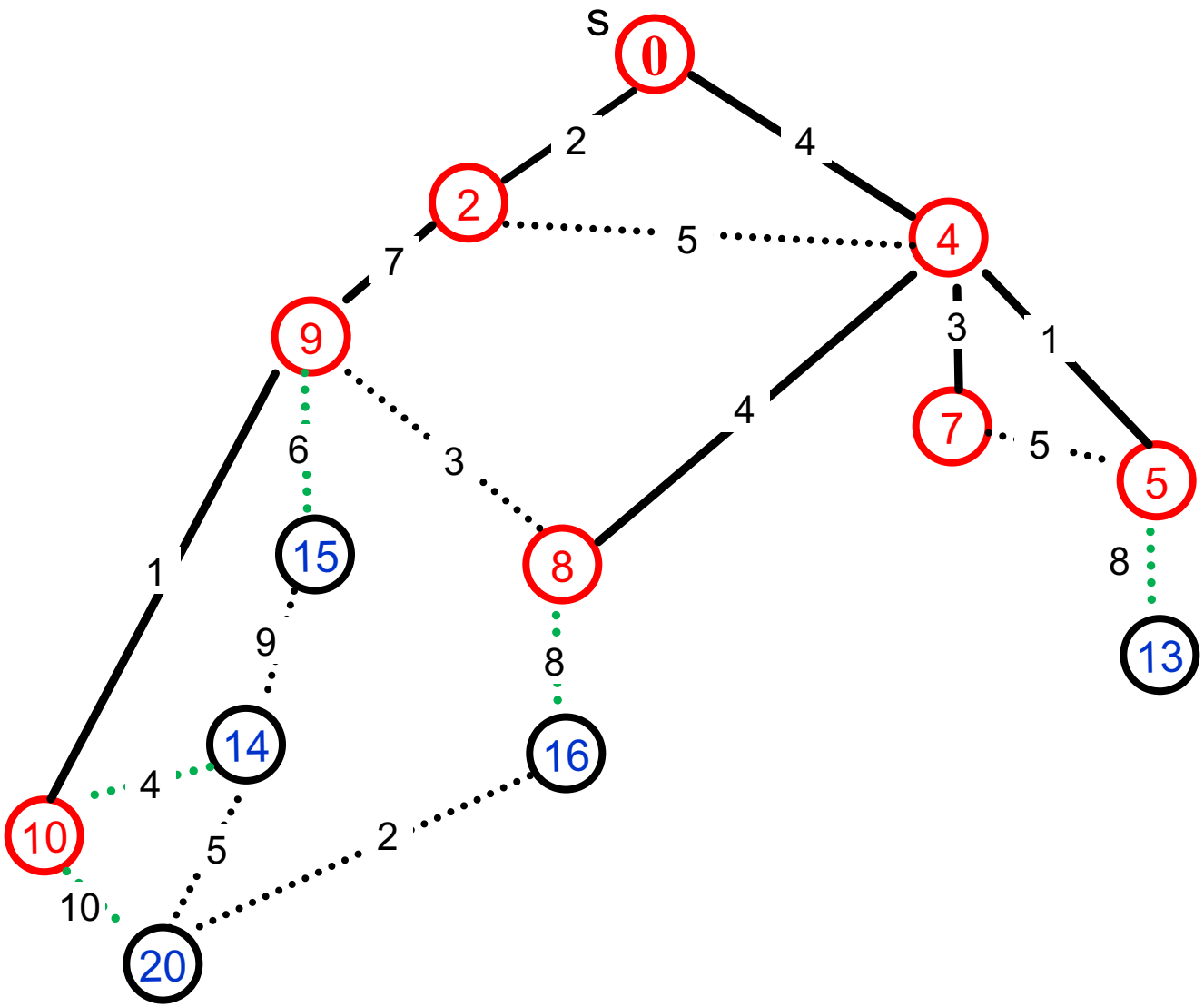

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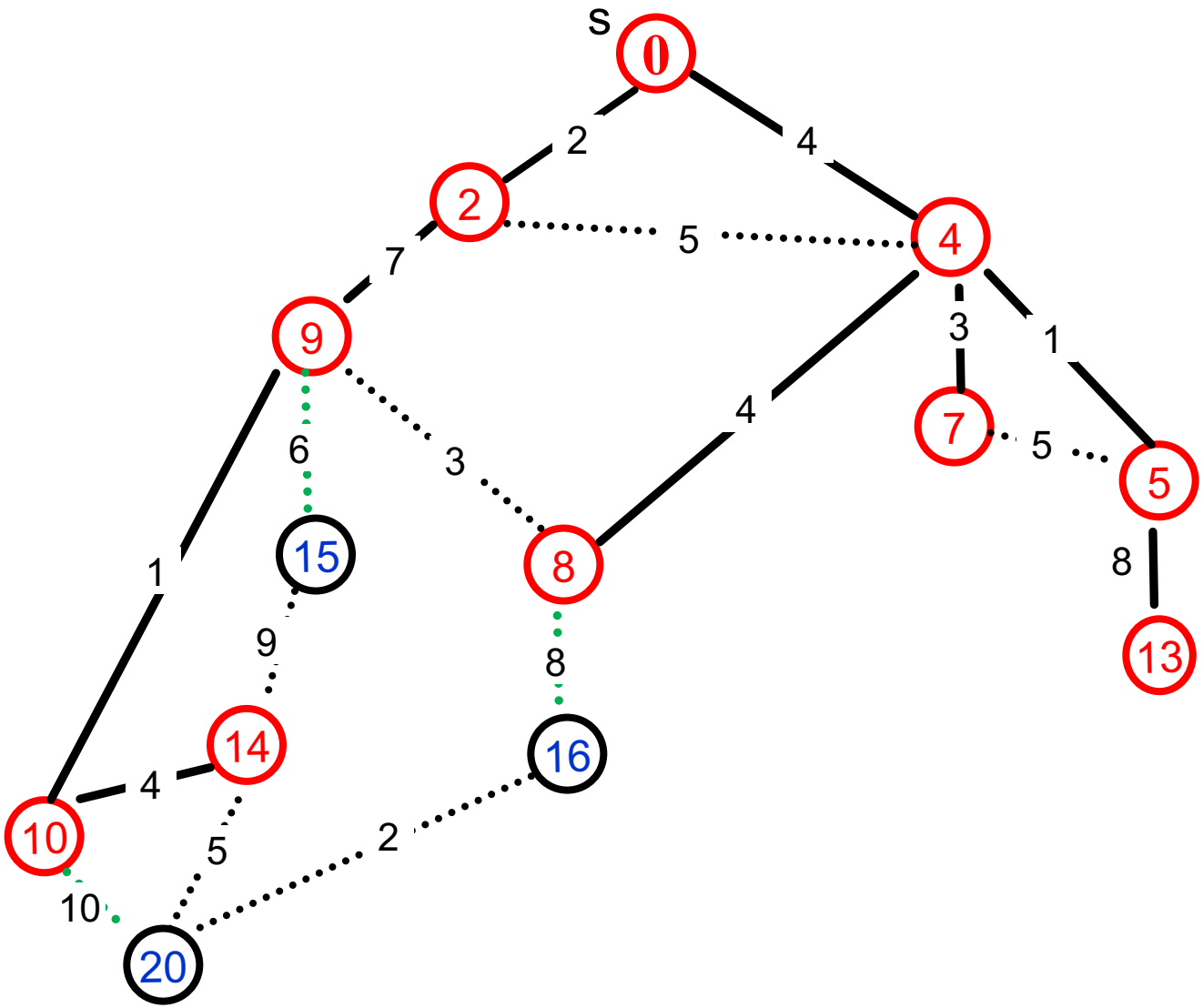

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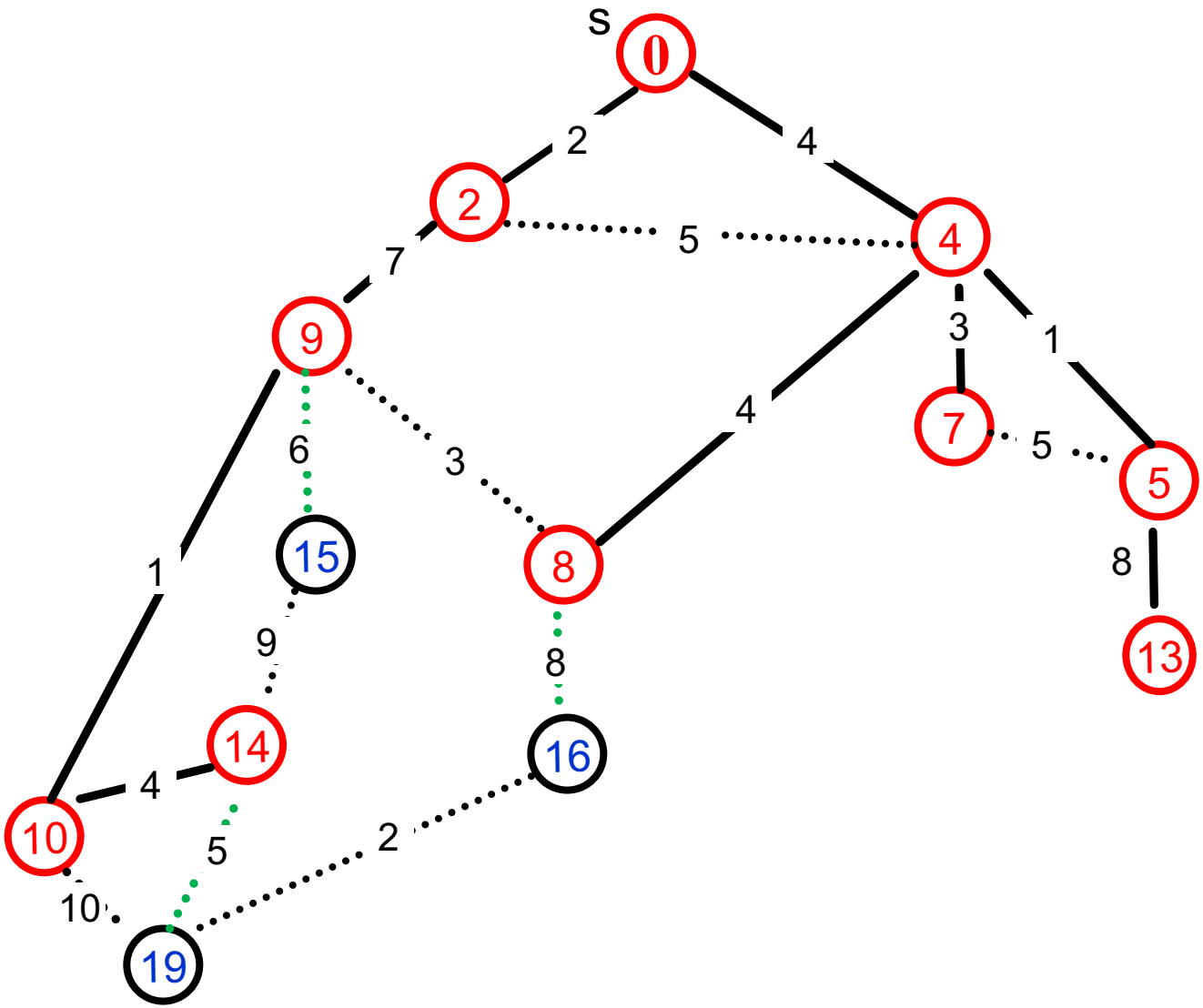

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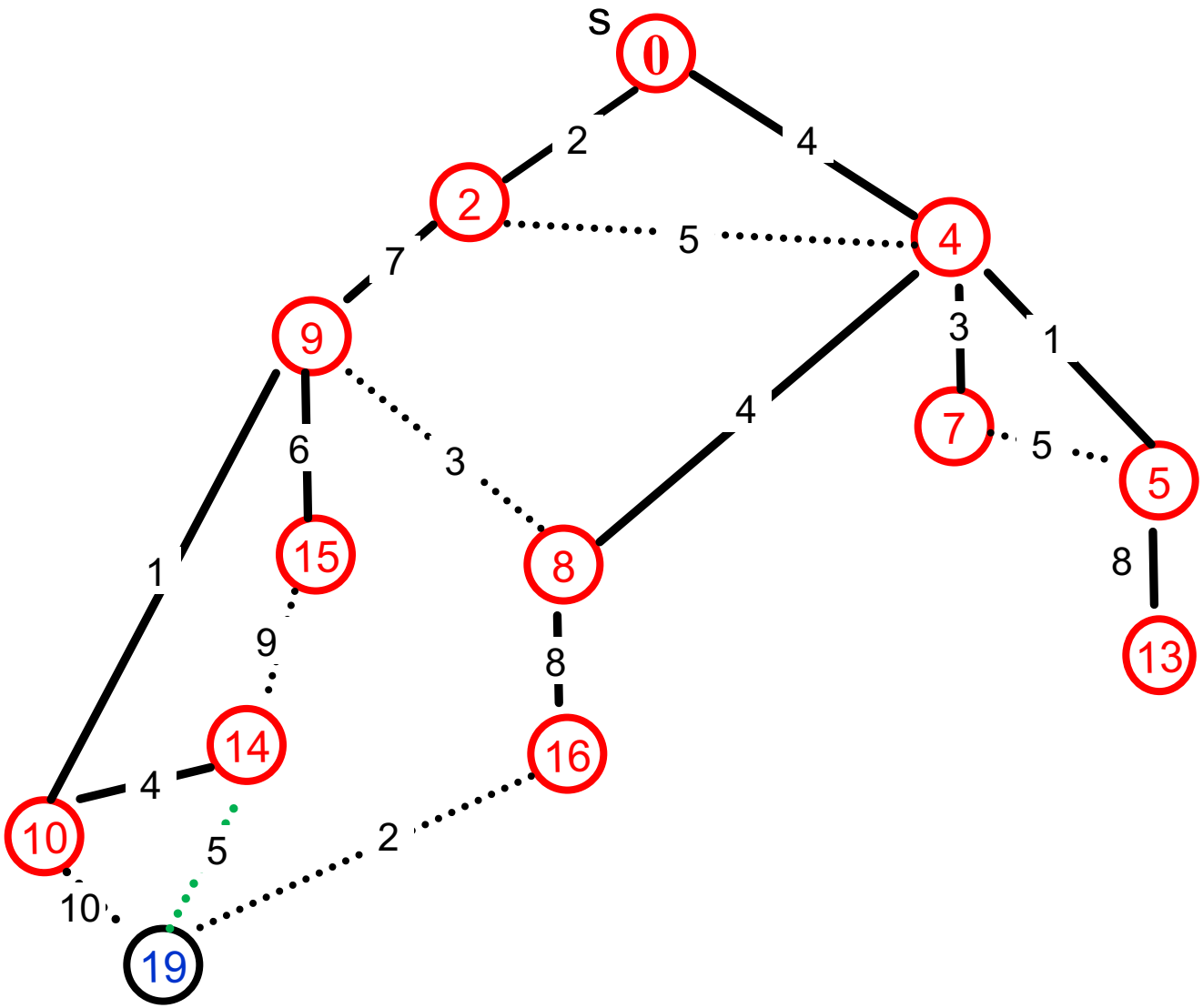

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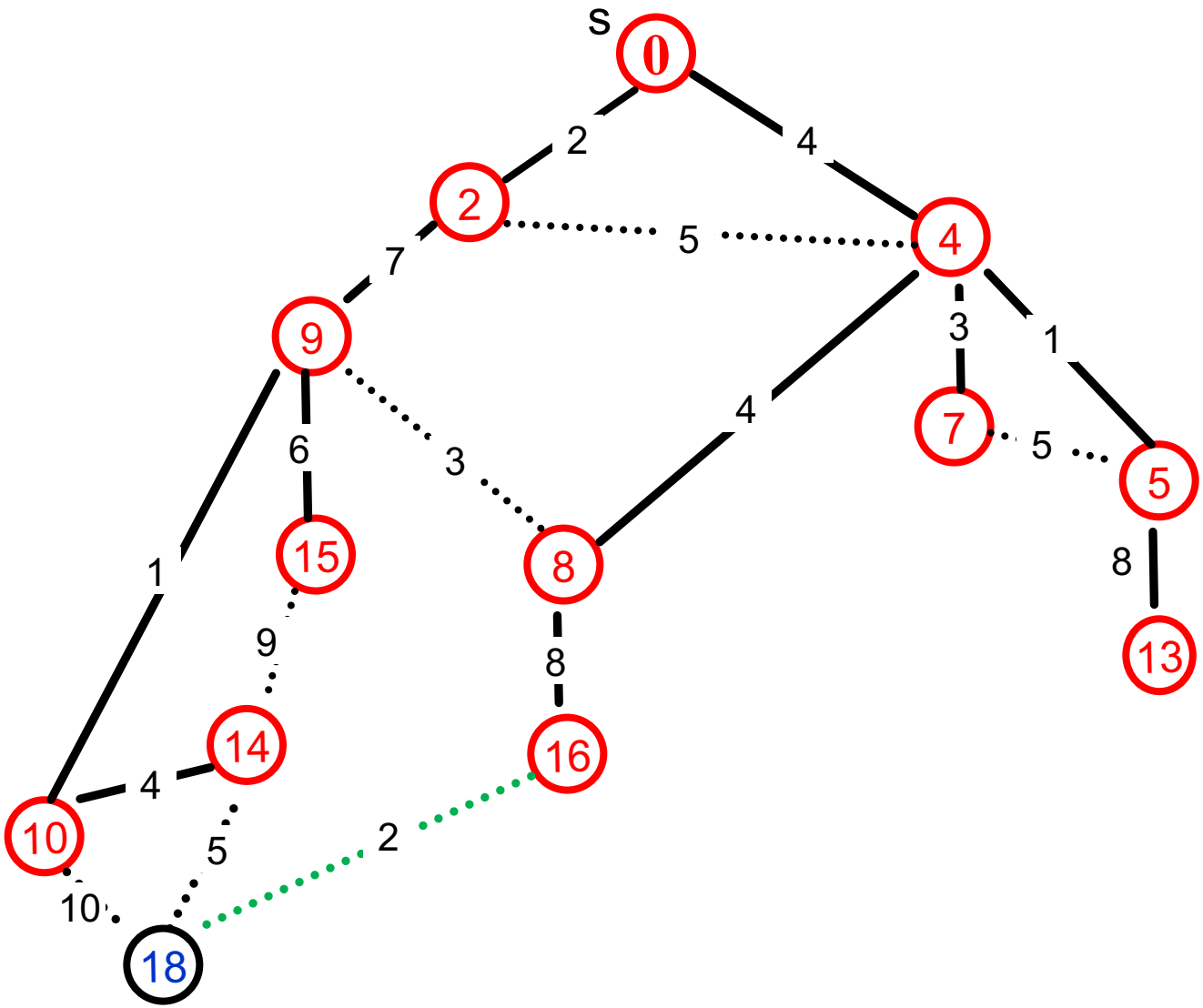

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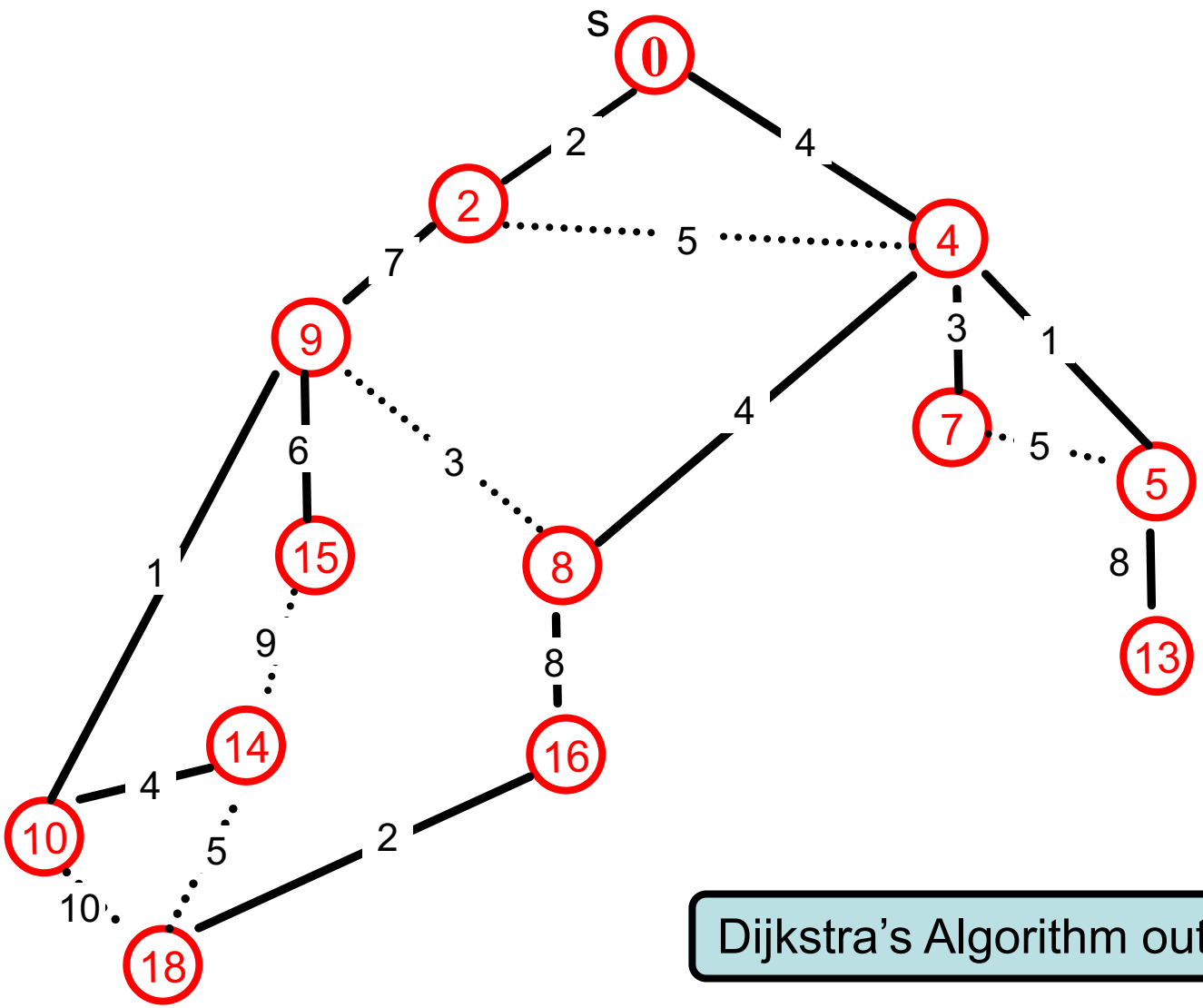


Dijkstra's Algorithm outputs a tree.

Disjkstra's Algorithm: Correctness

Theorem: For any $u \in S$, the path P_u on the tree in the shortest path from s to u on G. (For all $u \in S, d(u) = \text{dist}(s,u)$.)

Proof: Induction on $|S| = k$.

Base Case: This is always true when $S = \{s\}$.

Inductive Step: Say v is the $(k+1)^{st}$ vertex that we add to S.

Let (u,v) be last edge on P_v.

If P_v is not the shortest path, there is a shorter path P to S.

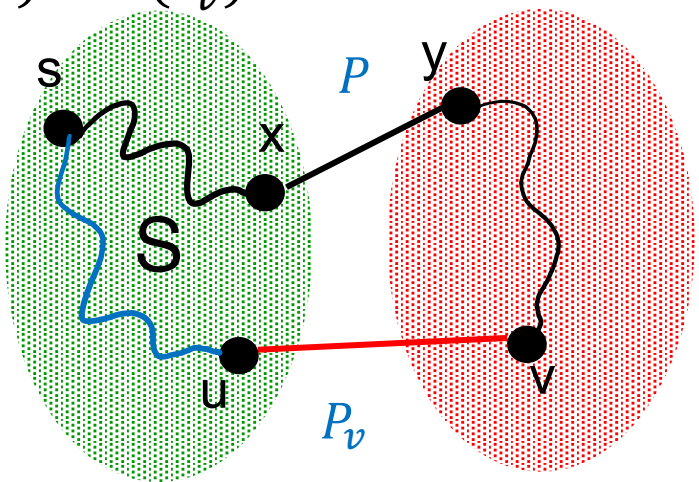
Consider the first time that P leaves S with edge (x,y).

So, $c(P) \geq d(x) + c_{x,y} \geq d(u) + c_{u,v} = d(v) = c(P_v)$.

P is the shorter path.

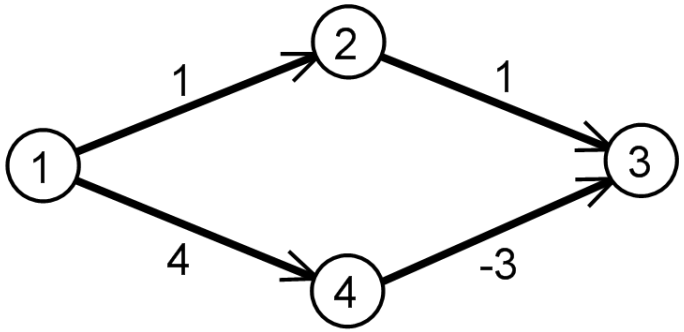
Due to the choice of v

A contradiction.



Remarks on Dijkstra's Algorithm

- Algorithm produces a tree of shortest paths to s following Parent links (for undirected graph)
- Algorithm works on directed graph (with nonnegative weights)
- The algorithm fails with negative edge weights.
- Why does it fail?



Implementing Dijkstra's Algorithm

Priority Queue: Elements each with an associated key Operations

- Insert
- Find-min
  - Return the element with the smallest key
- Delete-min
  - Return the element with the smallest key and delete it from the data structure
- Decrease-key
  - Decrease the key value of some element

Implementations

Arrays:

- $O(n)$ time find/delete-min,
- $O(1)$ time insert/decrease key

Binary Heaps:

- $O(\log n)$ time insert/decrease-key/delete-min,
- $O(1)$ time find-min

Fibonacci heap:

- $O(1)$ time insert/decrease-key
- $O(\log n)$ delete-min
- O(1) time find-min

Read wiki!

```
Dijkstra(G, c, s) {
    Initialize set of explored nodes S ← {s}

    // Maintain distance from s to each vertices in S
    d[s] ← 0
    Insert all neighbors v of s into a priority queue with value c_(s,v).

    while (S ≠ V)
    {
        Pick an edge (u, v) such that u ∈ S and v ∉ S and
        d[u] + c_(u,v) is as small as possible.
        v ← delete min element from Q

        Add v to S and define d[v] = d[u] + c_(u,v).
        Parent(v) ← u.

        foreach (edge e = (v, w) incident to v)
            if (w ∉ S)
                if (w is not in the Q)
                    Insert w into Q with value d[v] + c_(v,w)
                else (the key of w > d[v] + c_(v,w))
                    Decrease key of v to d[v] + c_(v,w).
}
```

$O(n)$ of insert, each in $O(1)$

$O(n)$ of delete min, each in $O(\log n)$

$O(m)$ of decrease/insert key, each runs in $O(1)$