# CS 401: Computer Algorithm I

## Single Source Shortest Path / Minimum Spanning Tree

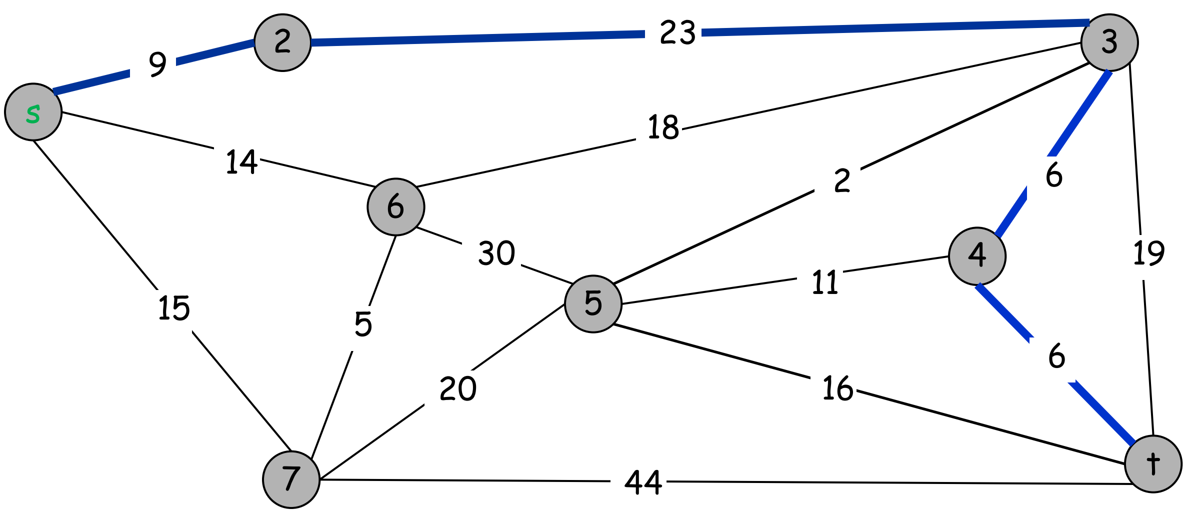
Xiaorui Sun

# Single Source Shortest Path

Given an (un)directed connected graph $G = (V, E)$ with non-negative edge weights $c_e \geq 0$ and a start vertex $s$.

Find length of shortest paths from $s$ to each vertex in $G$

length of path = sum of edge weights in path




Cost of path s-2-3-4-t
= 9 + 23 + 6 + 6
= 44.

Question: How to remove the assumption?

Relaxation: If we know the shortest paths from $s$ to all the other vertices with length smaller than $d[t]$, it is sufficient.

**Conclusion: Compute shortest paths in the ascending order of shortest path lengths**

Ob … n

fro … to

$v_k$.

Goal: compute the shortest path from $s$ to $t$

Assumption: know the shortest paths from $s$ to all the other vertices except $t$

$$d[t] = \min_{v \in V \setminus \{t\}} d[v] + c_{(v,t)}$$

($d[v]$ denotes the length of the shortest path from $s$ to $v$, and $c_{(v,t)}$ denotes the weight of the edge $(v, t)$)

# Dijkstra's Algorithm

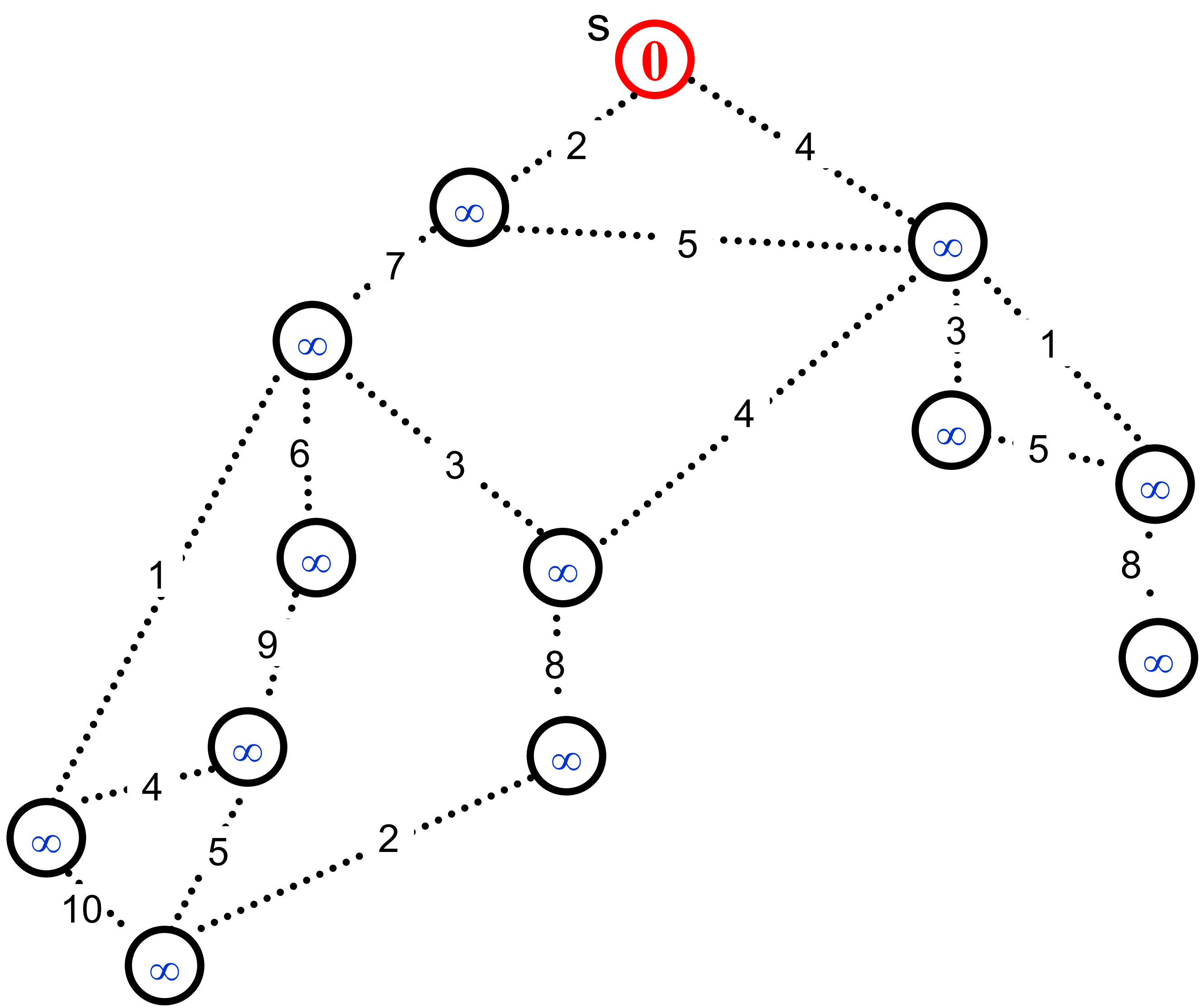
```
Dijkstra(G,c,s) {
   Initialize set of explored nodes S ← {s}

   // Maintain distance from s to each vertices in S
   d[s] ← 0

   while (S ≠ V)
   {
      Pick an edge (u,v) such that u ∈ S and v ∉ S and
            d[u] + c_(u,v) is as small as possible.

      Add v to S and define d[v] = d[u] + c_(u,v).
      Parent(v) ← u.
   }
```

# Dijkstra's Algorithm: Example



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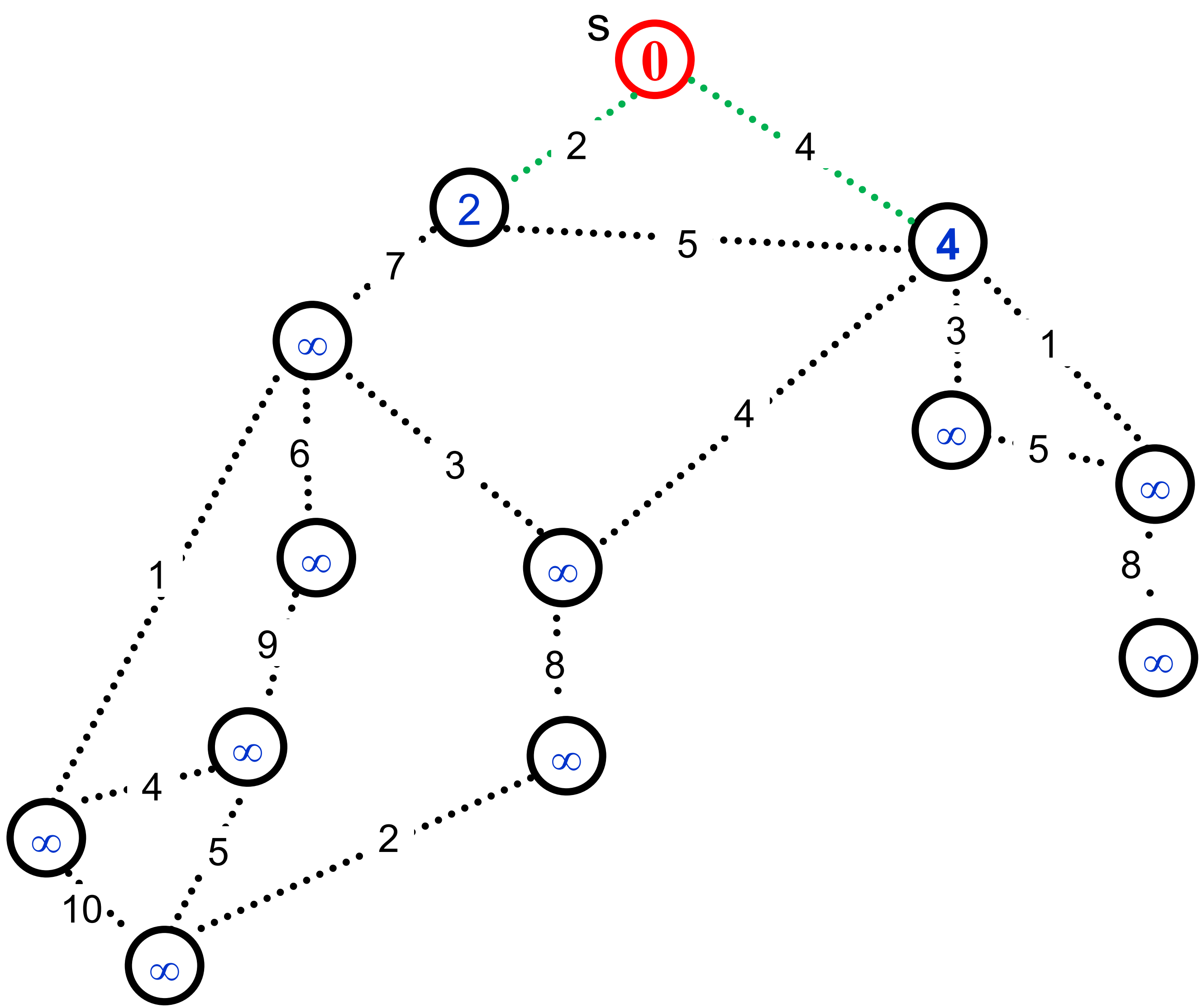

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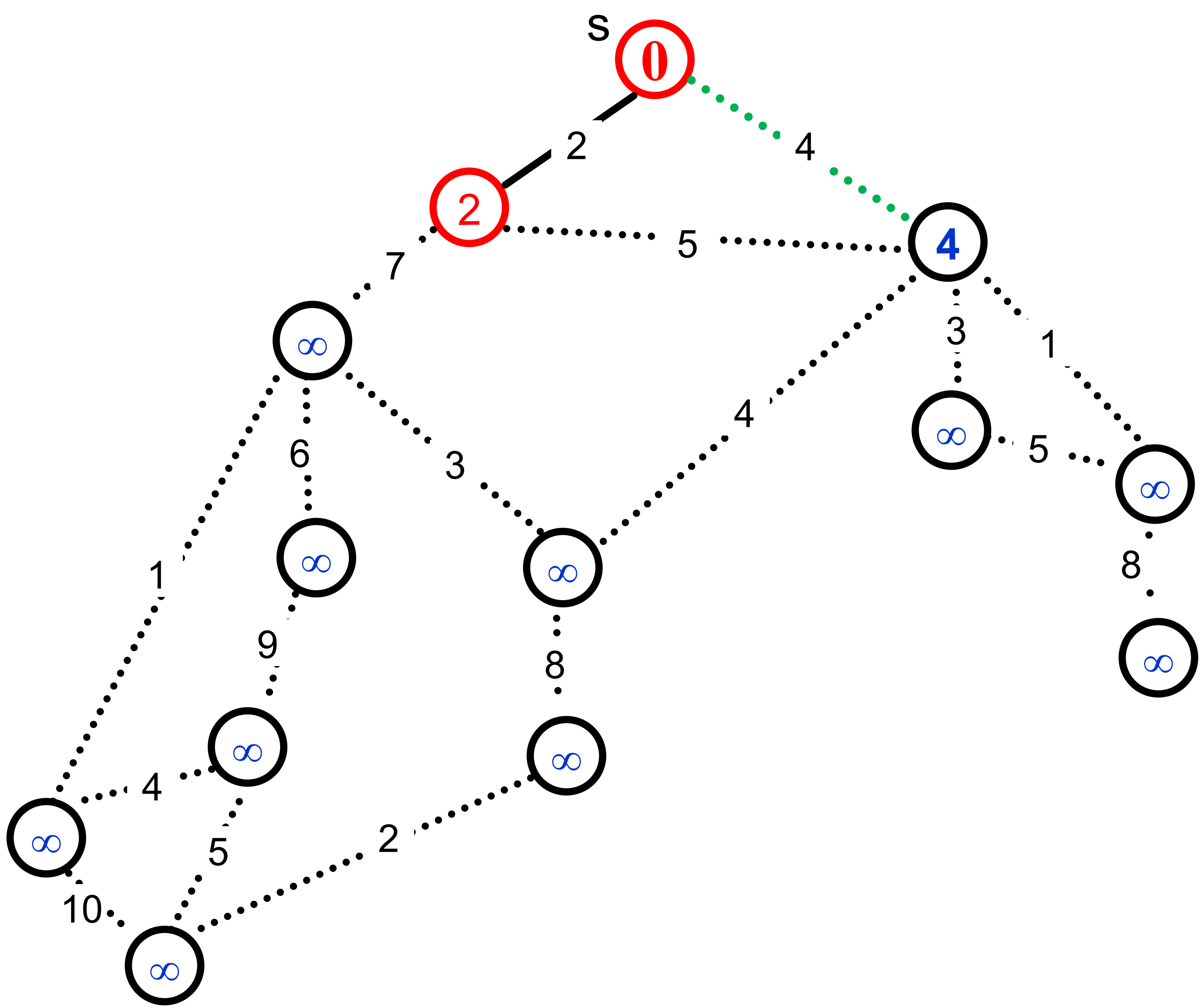

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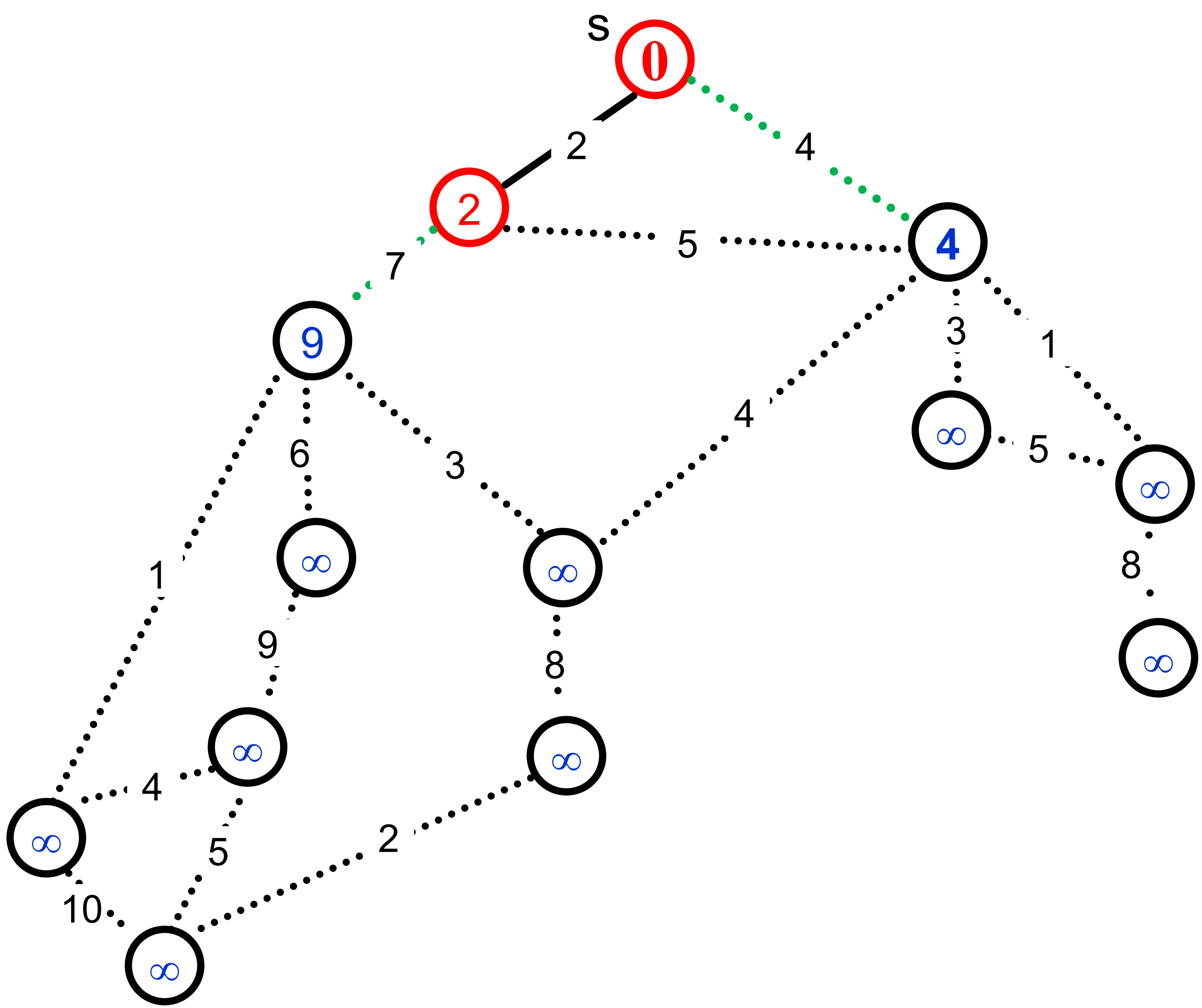

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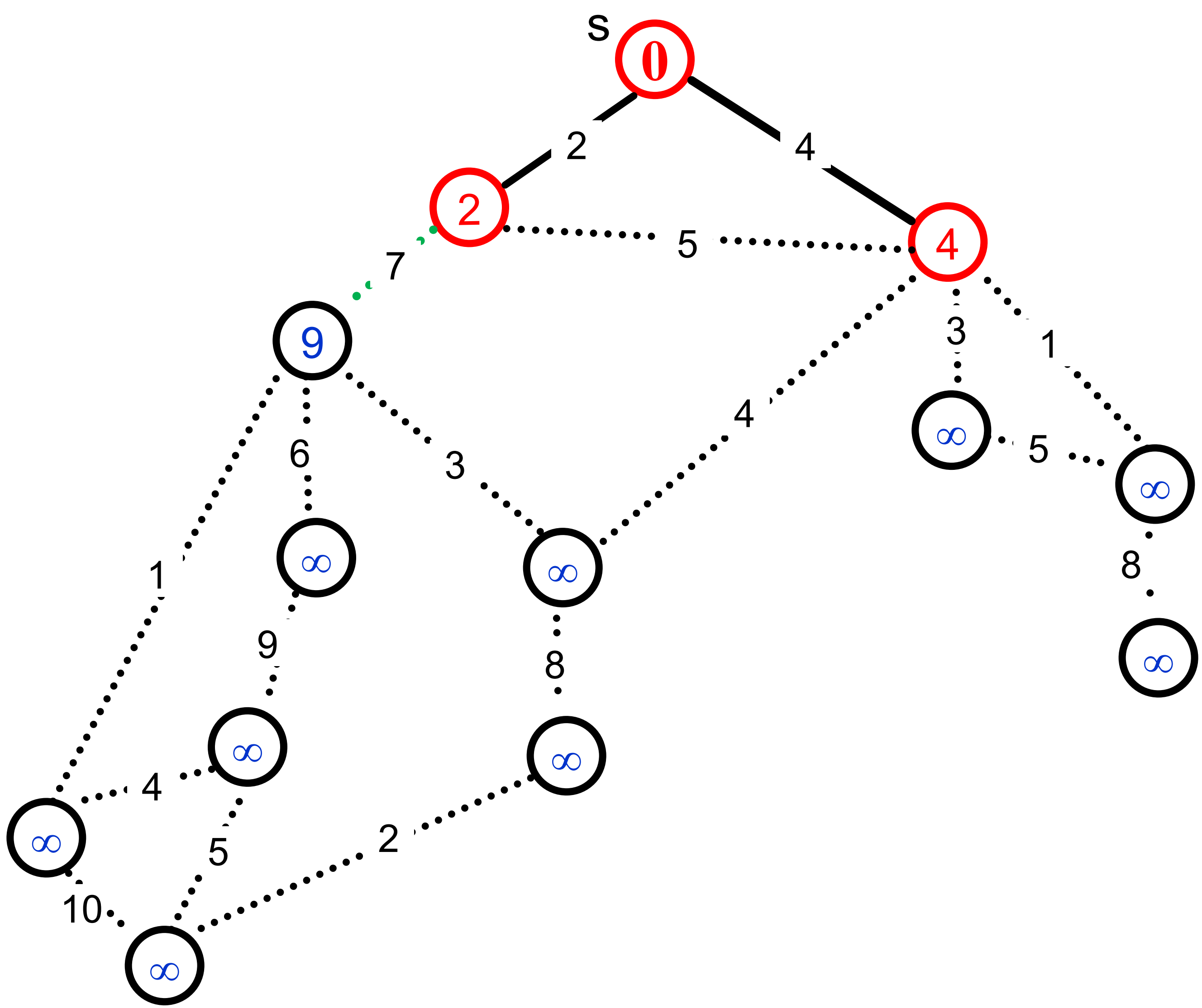

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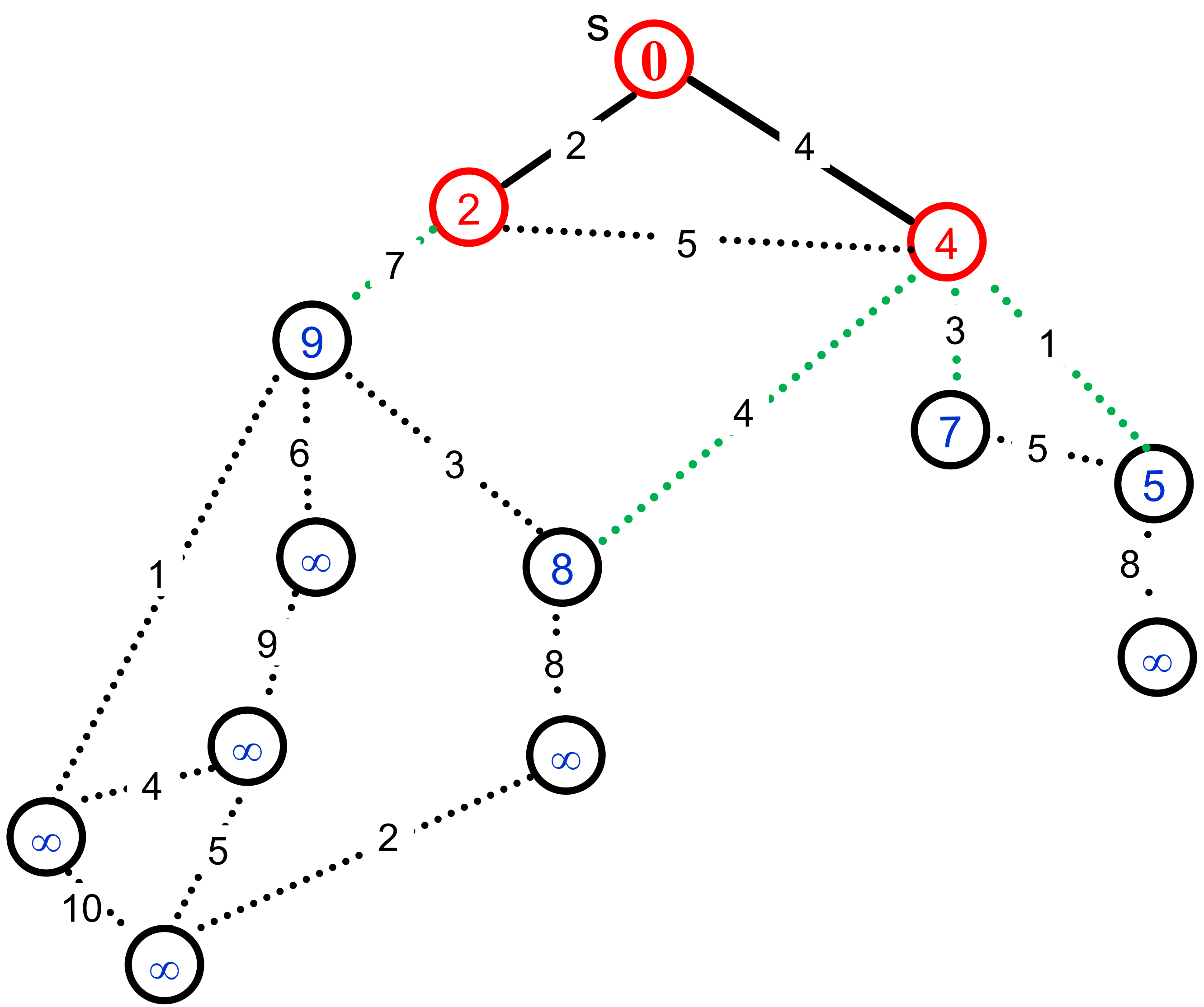

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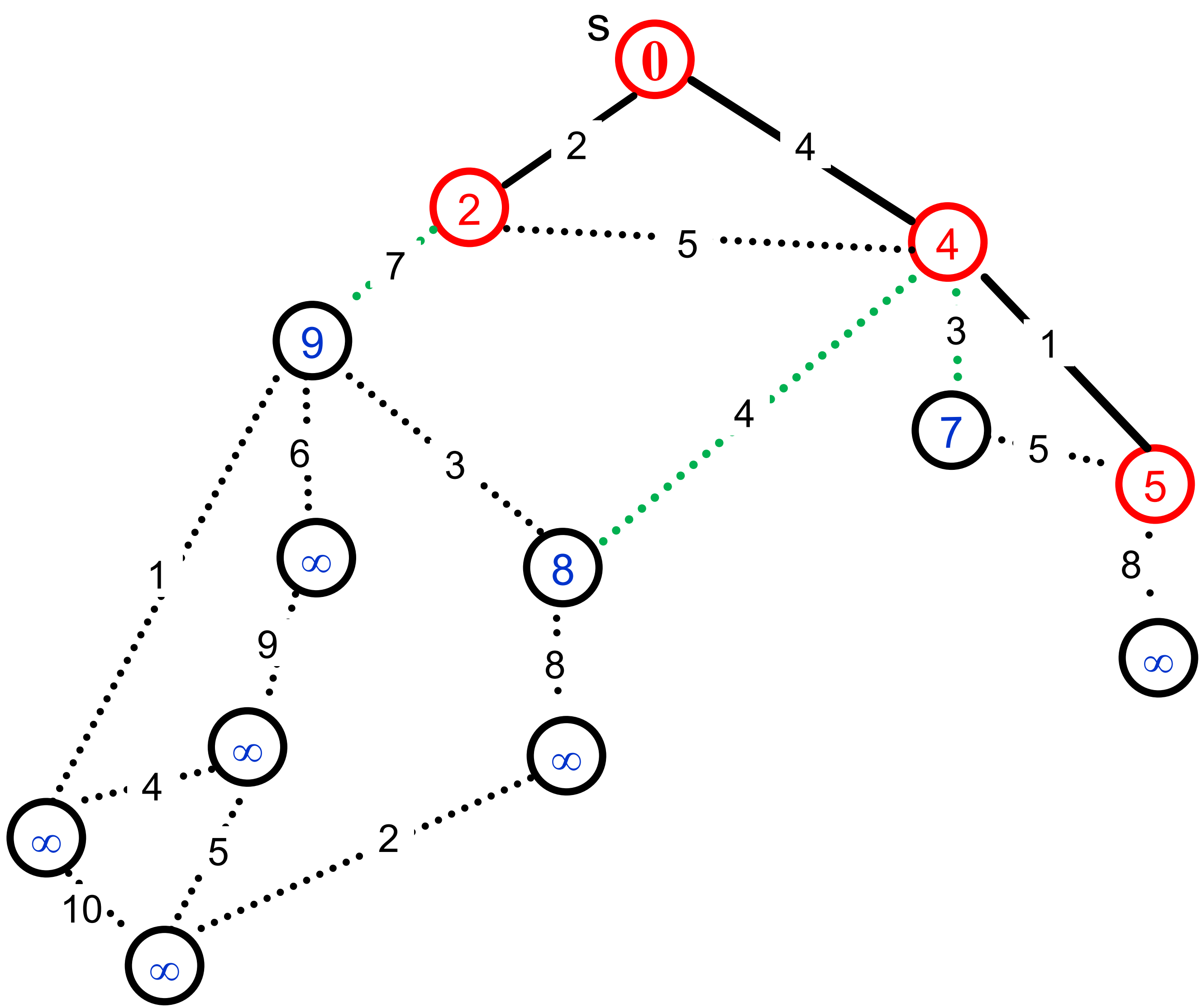

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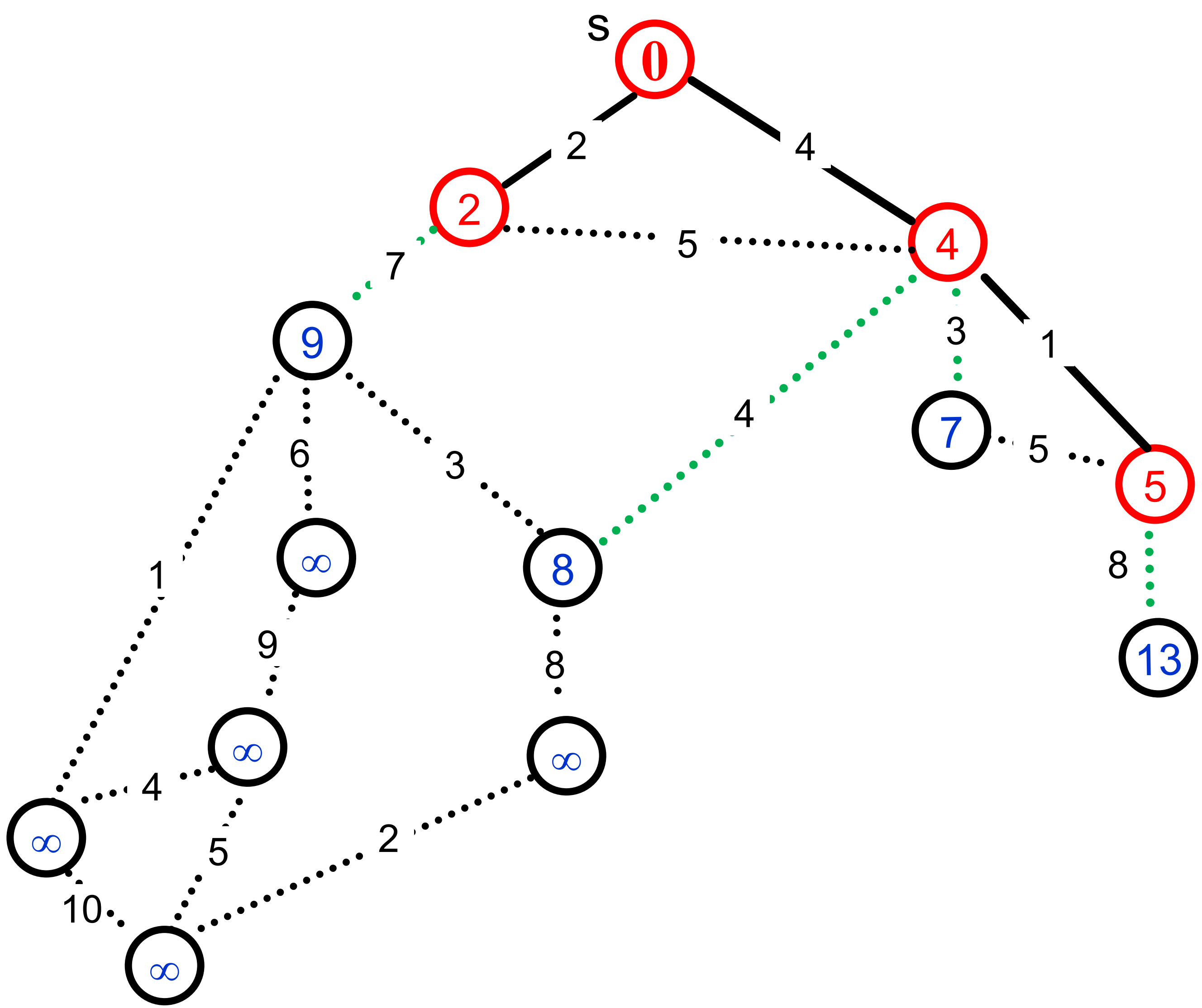

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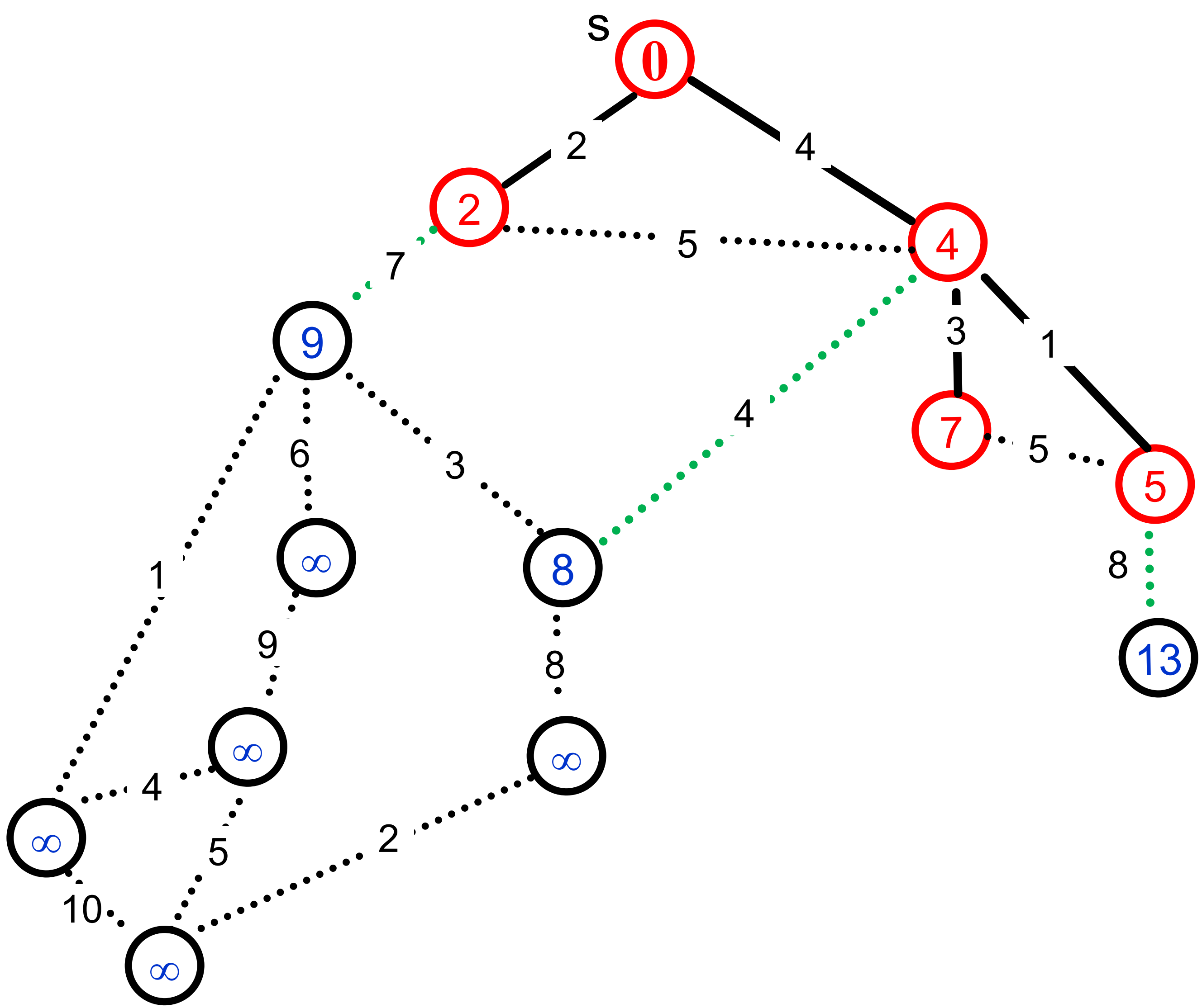

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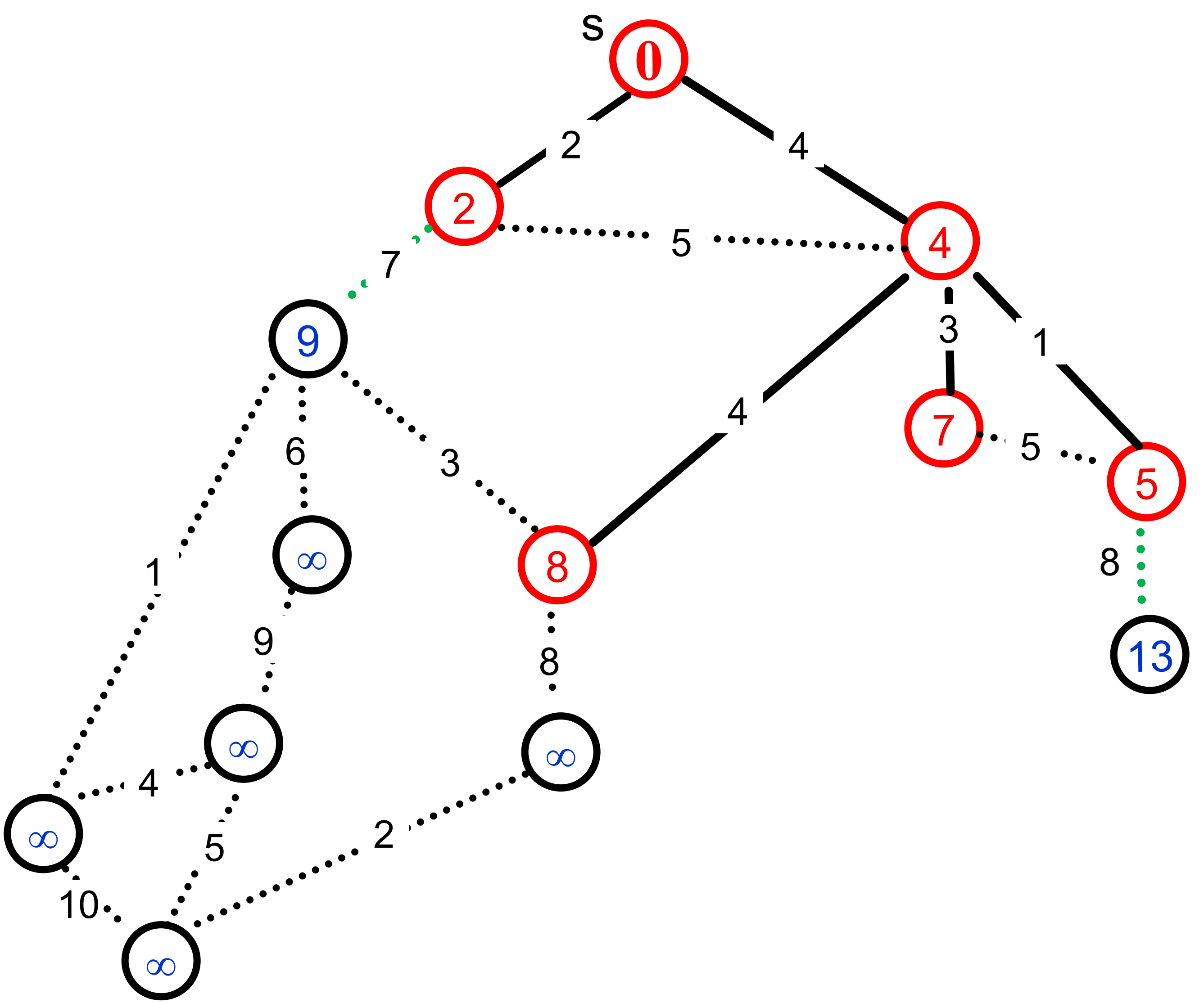

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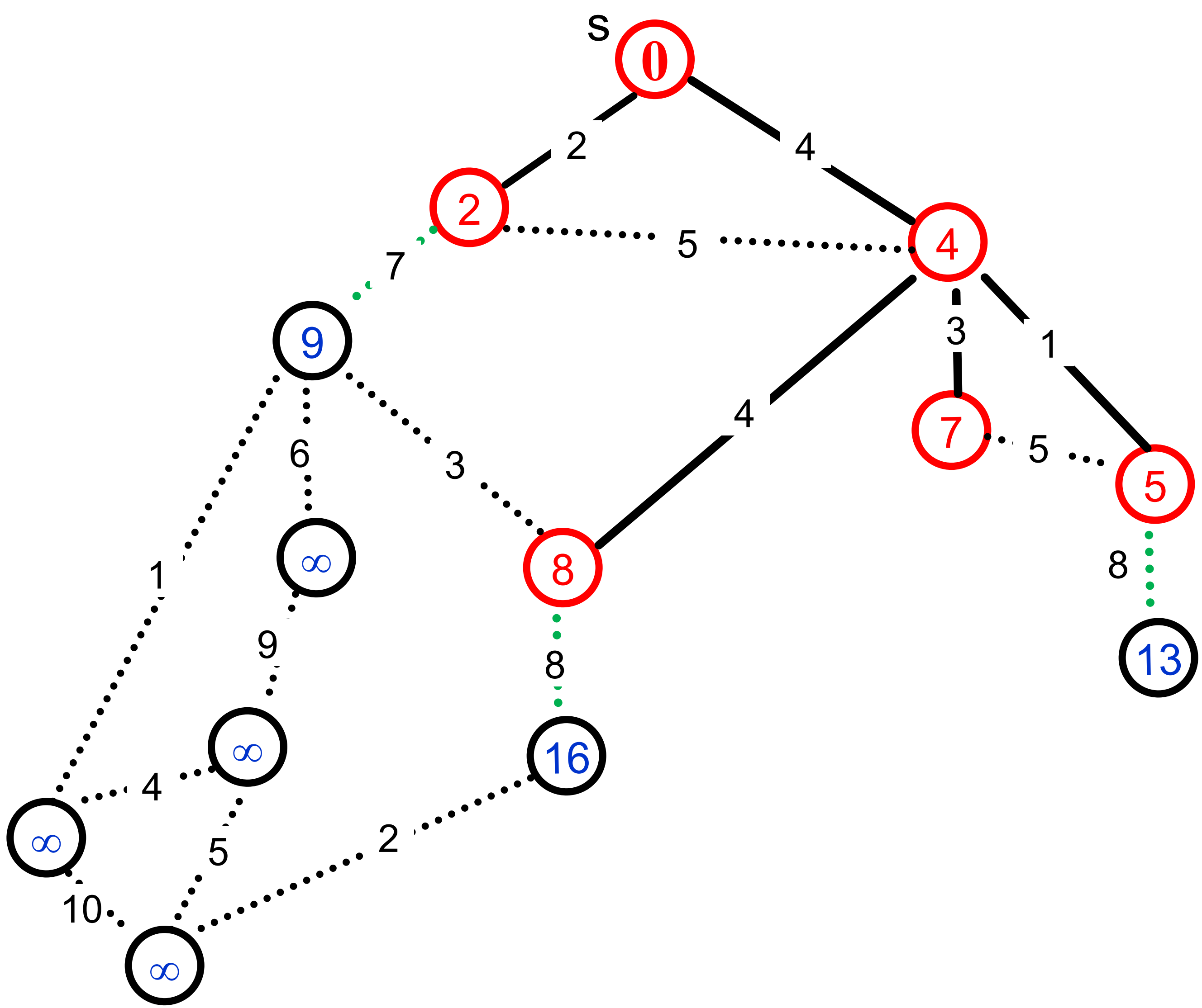

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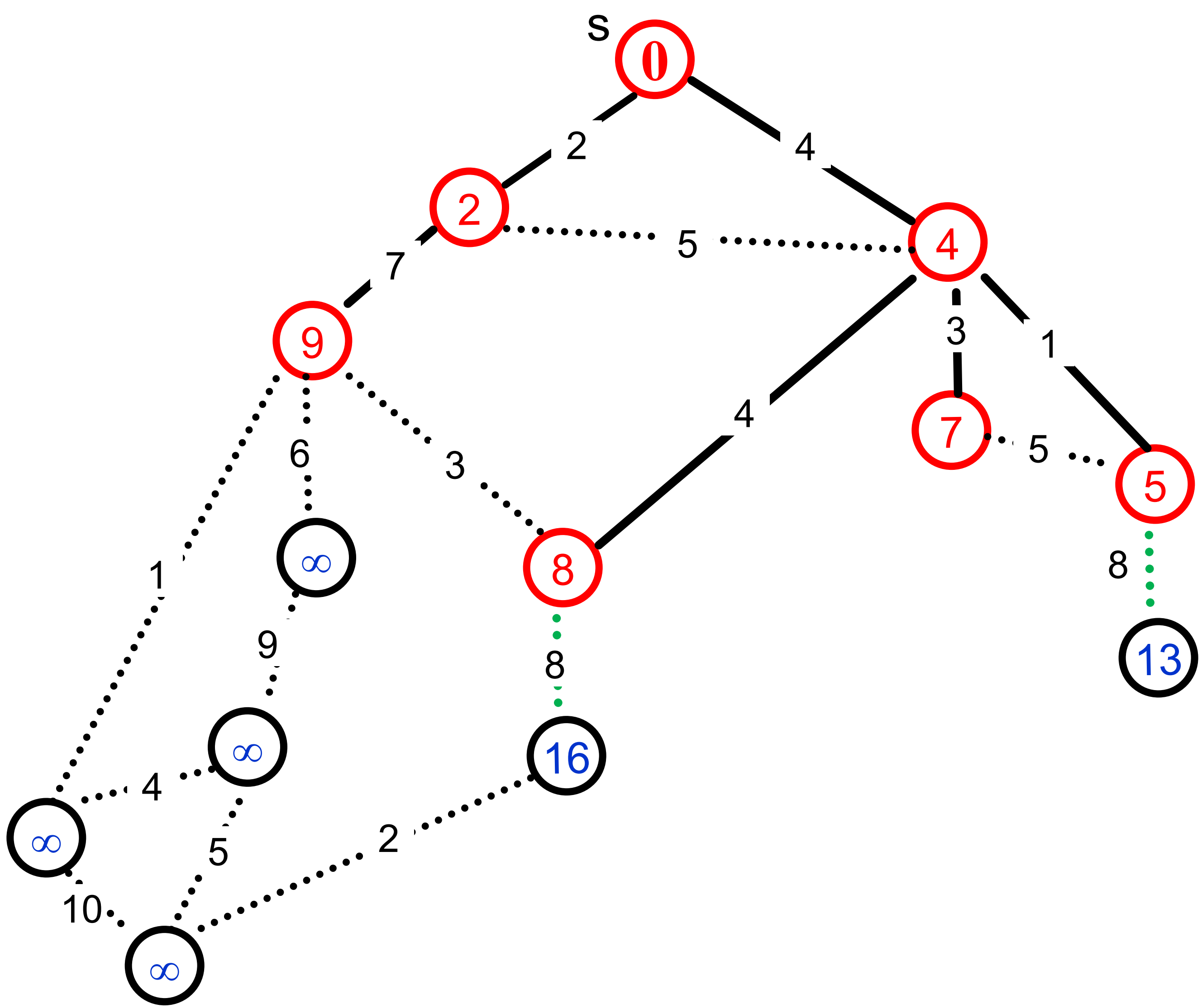

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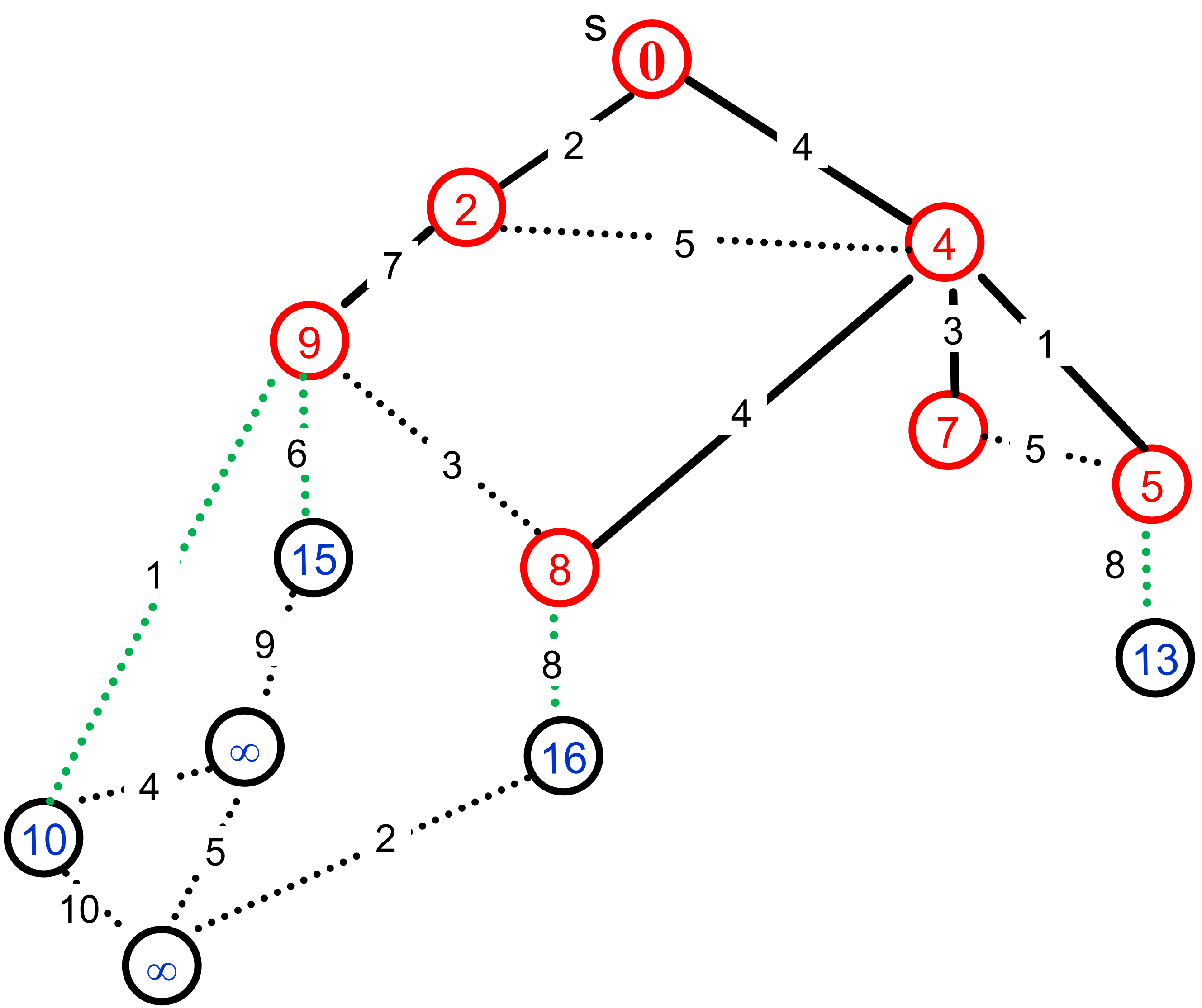

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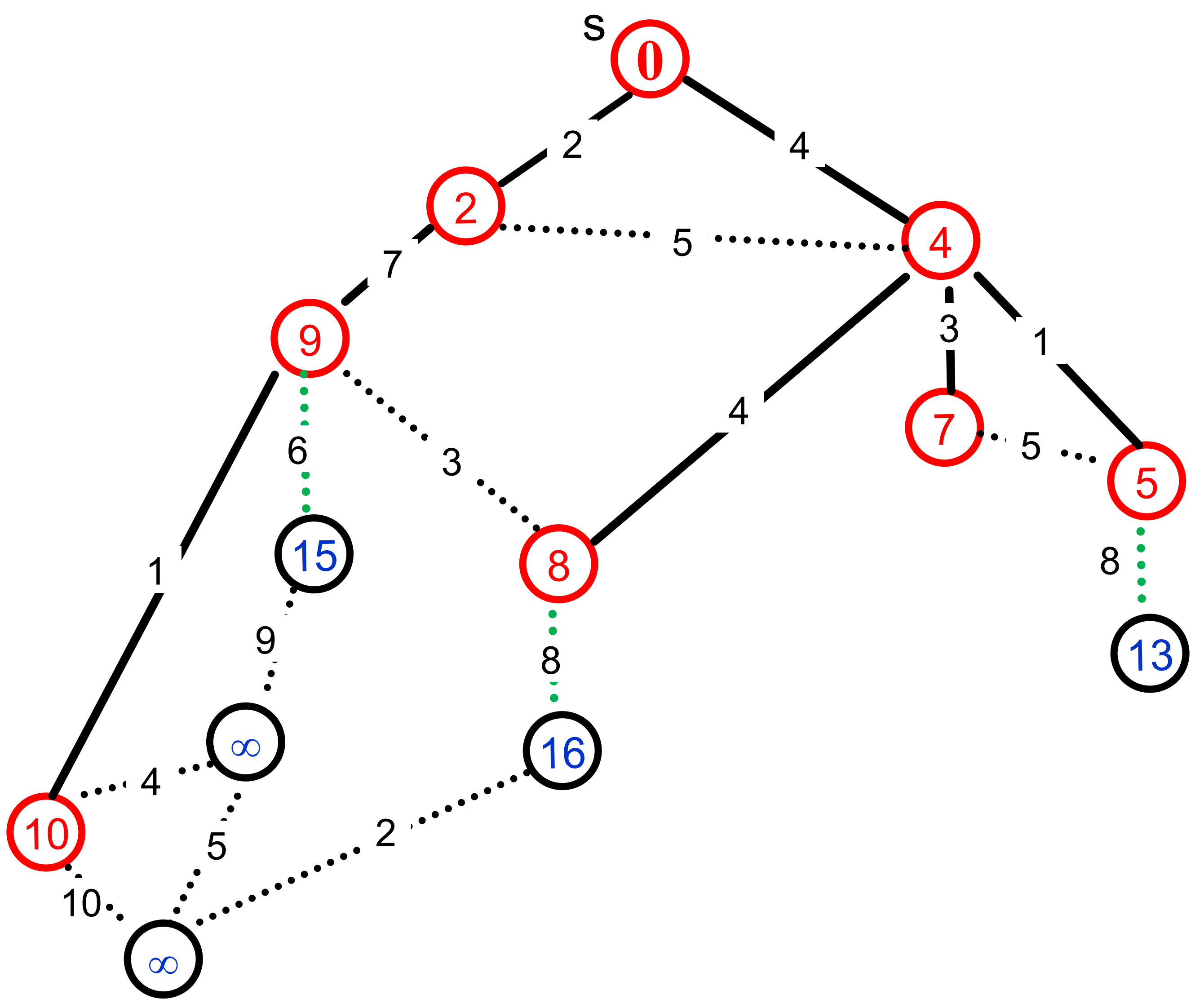

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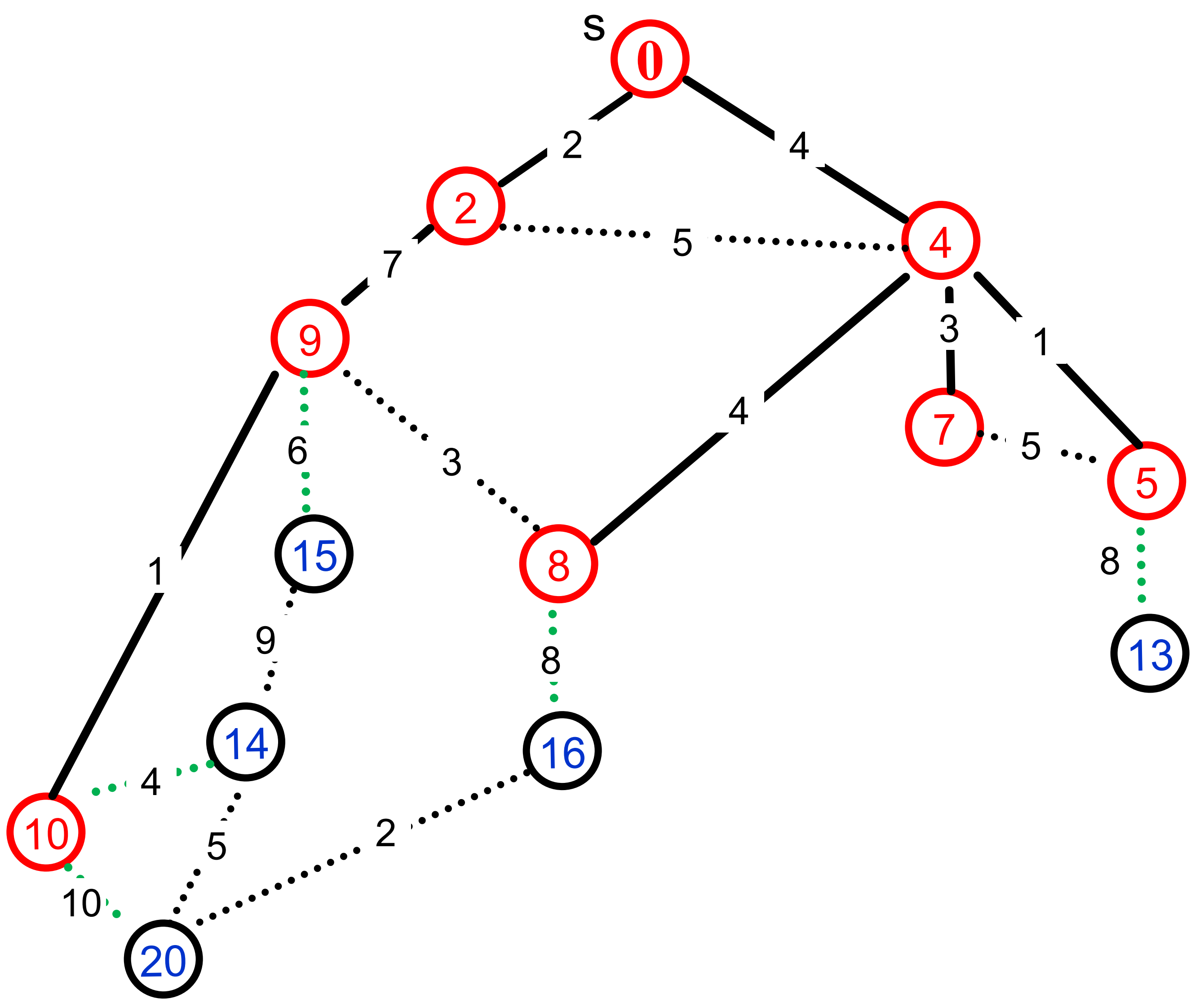

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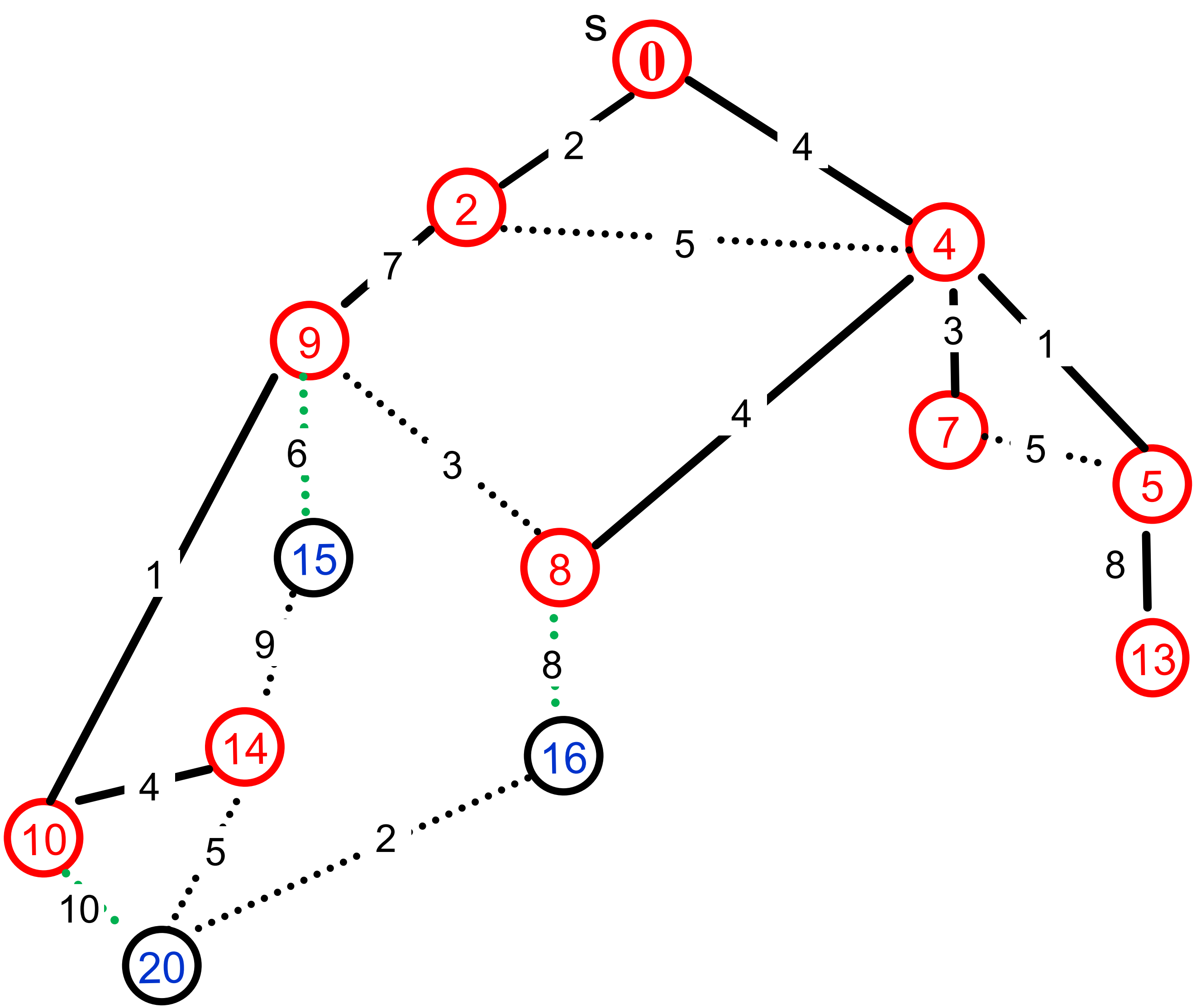

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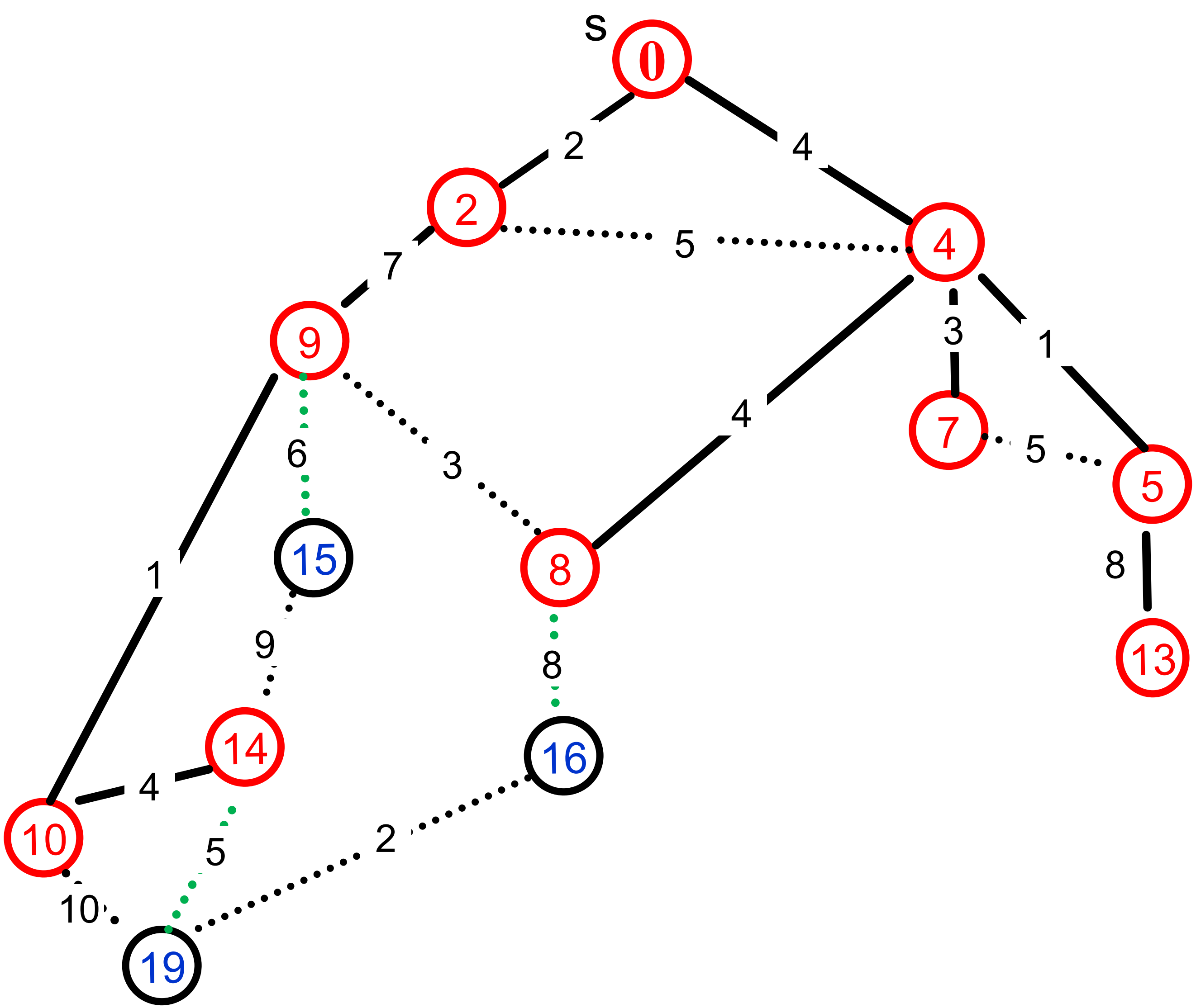

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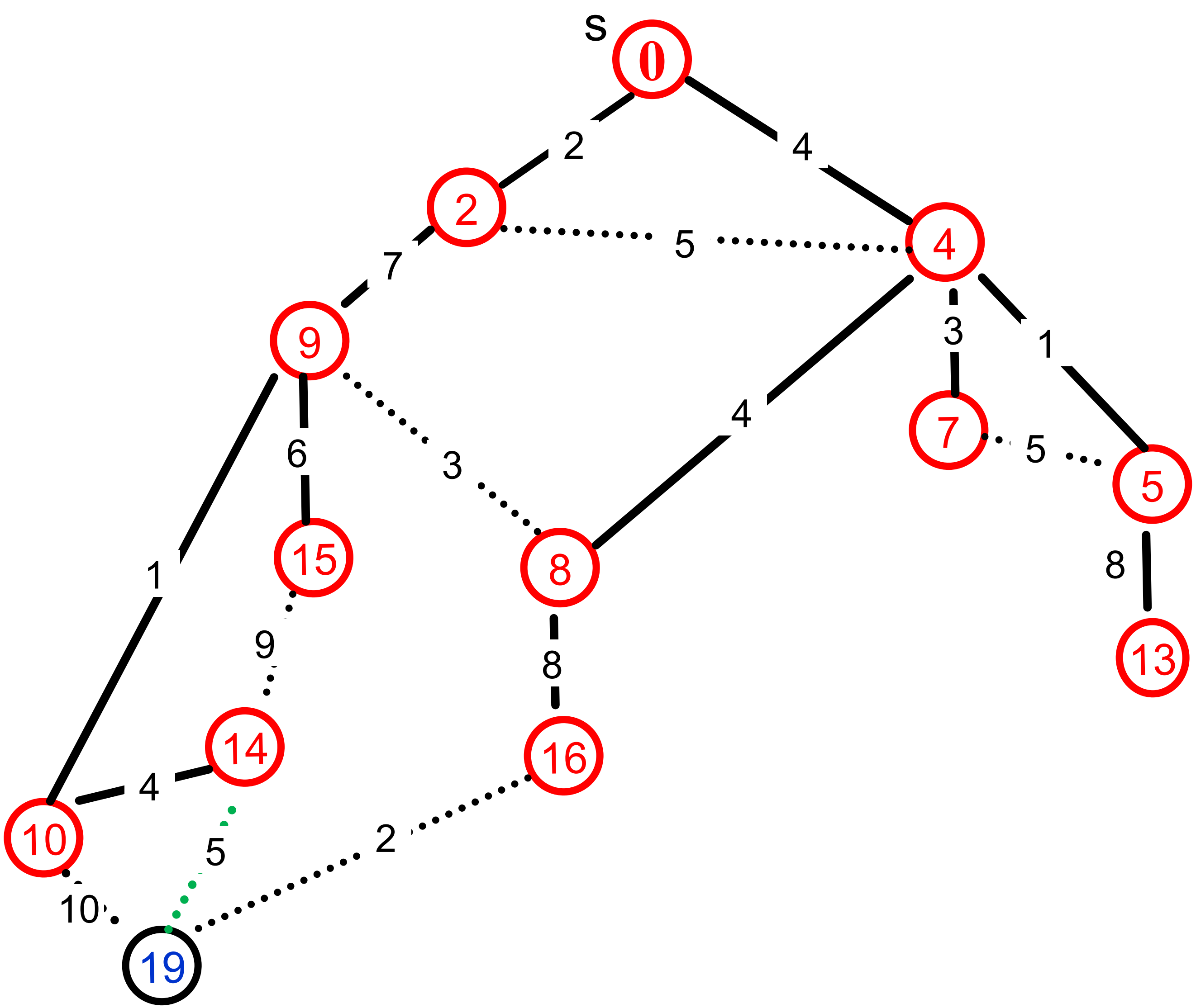

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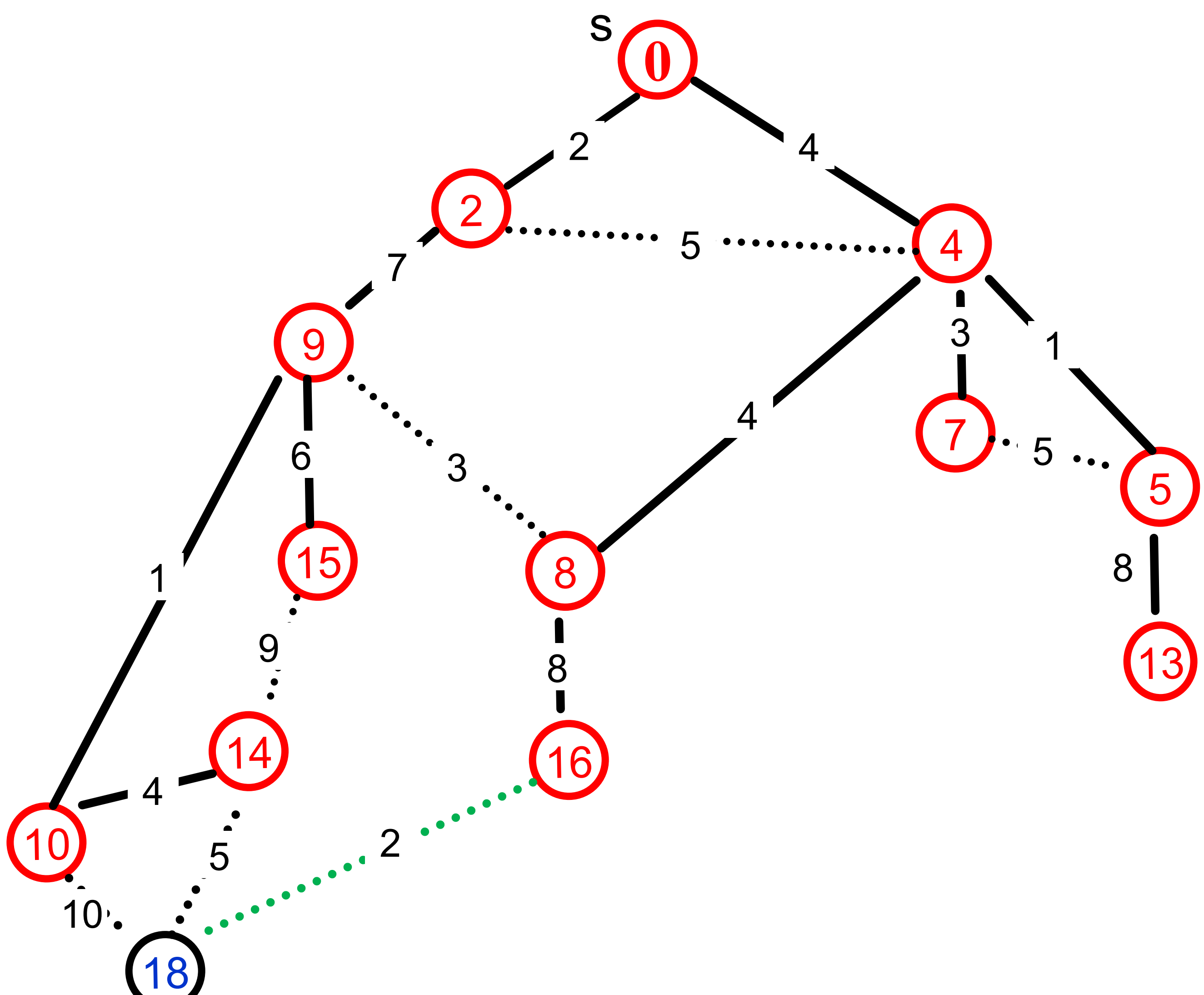

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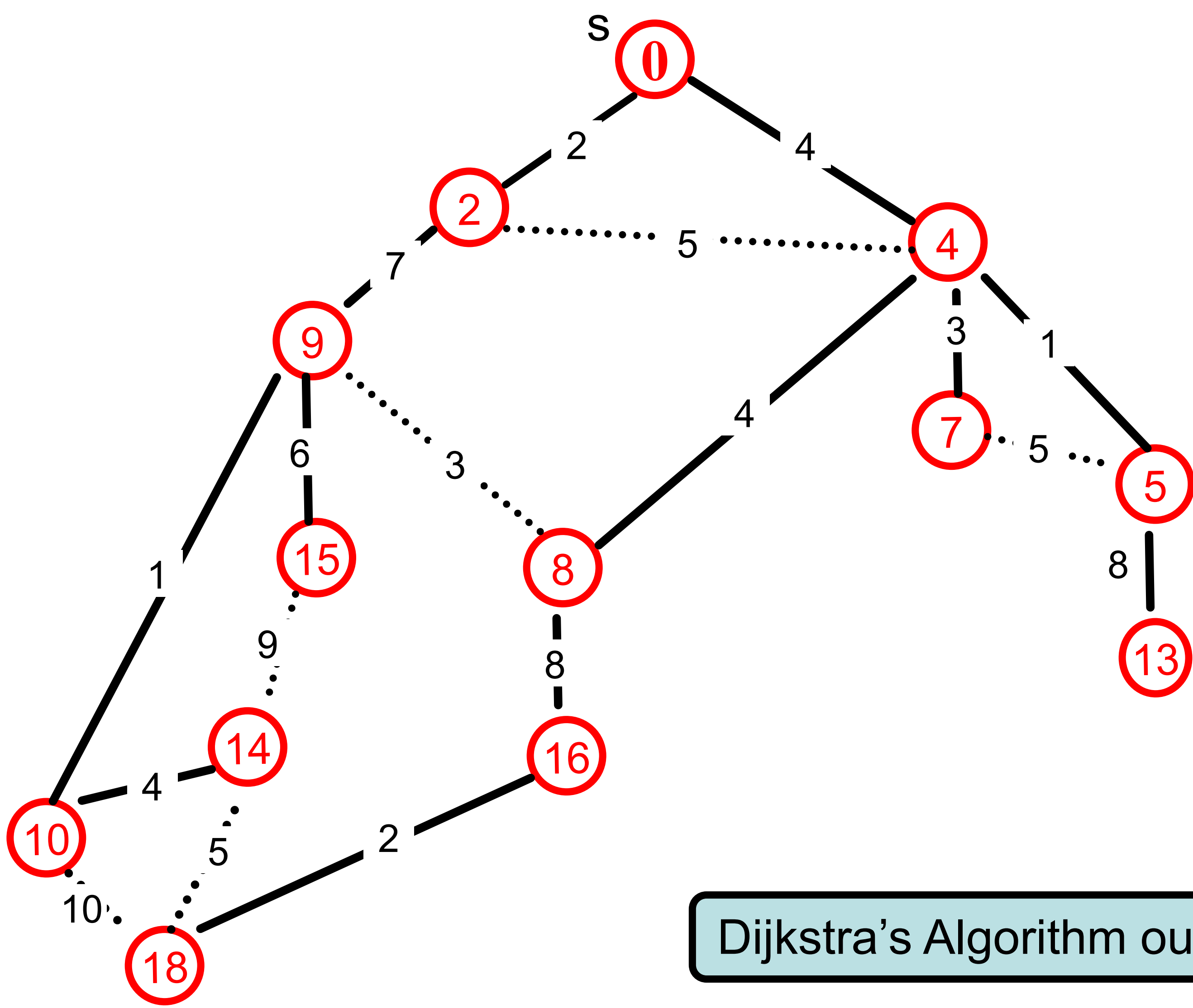



Dijkstra's Algorithm outputs a tree.

# Disjkstra's Algorithm: Correctness

Theorem: For any $u \in S$, the path $P_u$ on the tree in the shortest path from $s$ to $u$ on $G$. (For all $u \in S, d(u) = \mathrm{dist}(s, u)$.)

Proof: Induction on $|S| = k$.

Base Case: This is always true when $S = \{s\}$.

Inductive Step: Say $v$ is the $(k+1)^{st}$ vertex that we add to S.

Let $(u, v)$ be last edge on $P_v$.

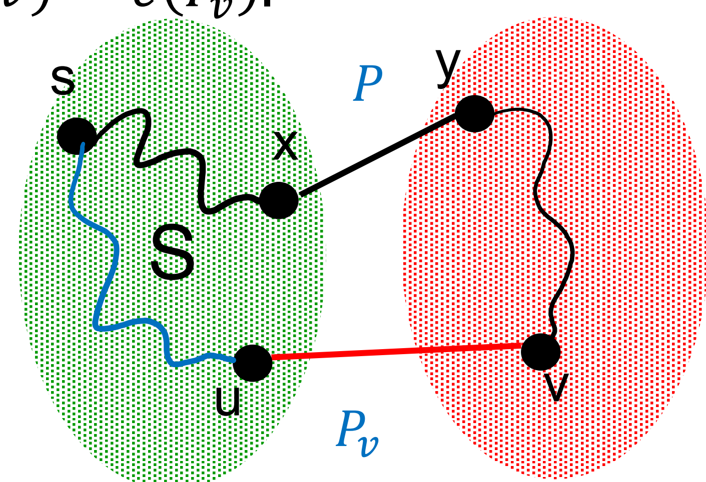
If $P_v$ is not the shortest path, there is a shorter path $P$ to $S$.

Consider the first time that $P$ leaves $S$ with edge $(x, y)$.

So, $c(P) \geq d(x) + c_{x,y} \geq d(u) + c_{u,v} = d(v) = c(P_v)$.

$P$ is the shorter path.

Due to the choice of $v$




A contradiction.

# Implementing Dijkstra's Algorithm

Priority Queue: Elements each with an associated key Operations

- Insert
- Find-min
  - Return the element with the smallest key
- Delete-min
  - Return the element with the smallest key and delete it from the data structure
- Decrease-key
  - Decrease the key value of some element

Implementations

Arrays:

- $O(n)$ time find/delete-min,
- $O(1)$ time insert/decrease key

$O(n^2 + m)$ time

Binary Heaps:

- $O(\log n)$ time insert/decrease-key/delete-min,
- $O(1)$ time find-min

$O(m \log n)$ time
Fast enough usually

Fibonacci heap:

- $O(1)$ time insert/decrease-key
- $O(\log n)$ delete-min
- O(1) time find-min

Read wiki!

$O(m + n \log n)$ time
Even faster theoretically

```
Dijkstra(G,c,s) {
   Initialize set of explored nodes S ← {s}

   // Maintain distance from s to each vertices in S
   d[s] ← 0
   Insert all neighbors v of s into a priority queue with value c_(s,v).

   while (S ≠ V)
   {
      Pick an edge (u,v) such that u ∈ S and v ∉ S and
      d[u] + c_(u,v) is as small as possible.
      v ← delete min element from Q

      Add v to S and define d[v] = d[u] + c_(u,v).
      Parent(v) ← u.

      foreach (edge e = (v,w) incident to v)
          if (w ∉ S)
             if (w is not in the Q)
                Insert w into Q with value d[v] + c_(v,w)
             else (the key of w > d[v] + c_(v,w))
                Decrease key of v to d[v] + c_(v,w).
}
```

$O(n)$ of insert, each in $O(1)$

$O(n)$ of delete min, each in $O(\log n)$

$O(m)$ of decrease/insert key, each runs in $O(1)$

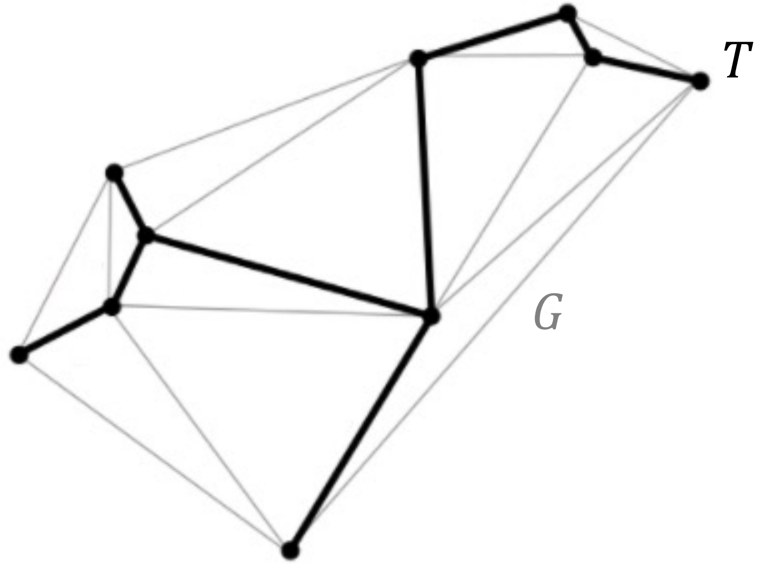
# Minimum Spanning Tree

# Spanning Tree

Given a connected undirected graph $G = (V, E)$.

We call $T$ is a spanning tree of $G$ if

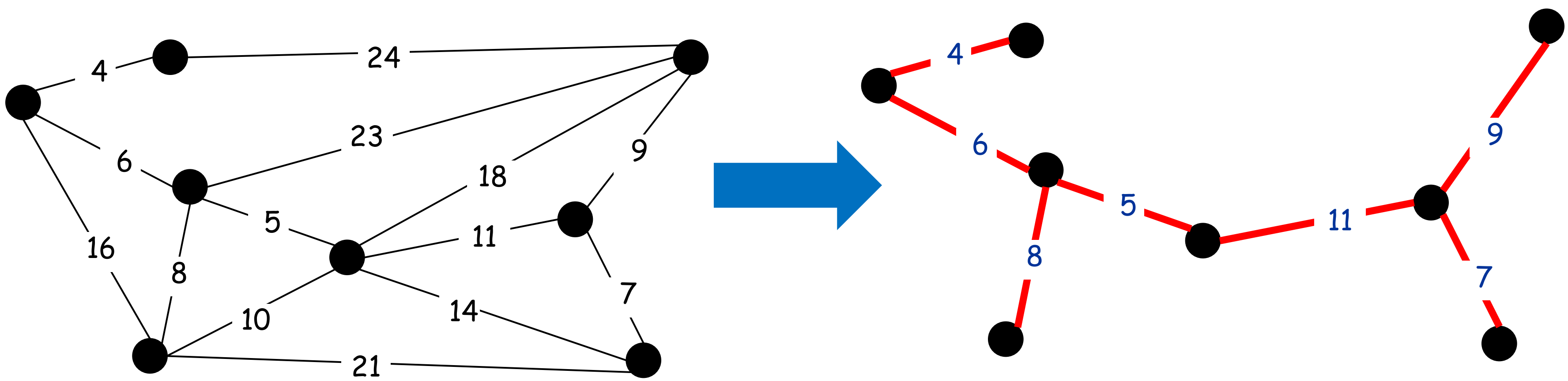
- All edges in $T$ are from $E$.
- $T$ includes all of the vertices of $G$.



# Minimum Spanning Tree (MST)

Given a connected undirected graph $G = (V, E)$ with real-valued edge weights $c_e \geq 0$.

An MST $T$ is a spanning tree whose sum of edge weights is minimized.



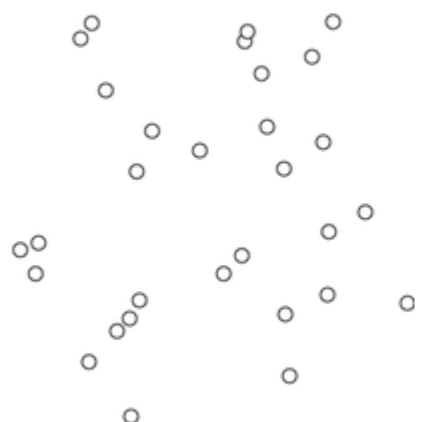

$$G = (V, E)$$

$$c(T) = \sum_{e \in T} c_e = 50$$

# Kruskal's Algorithm [1956]

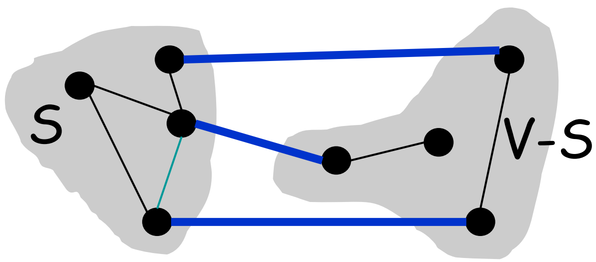
Kruskal(G, c) {<br>
    Sort edges weights so that $c_1 \le c_2 \le \cdots \le c_m$.<br>
    $T \leftarrow \emptyset$<br>
<br>
    foreach $(u \in V)$ make a set containing singleton $\{u\}$<br>
<br>
    for $i = 1$ to $m$<br>
        Let $(u, v) = e_i$<br>
        if ($u$ and $v$ are in different sets) {<br>
            $T \leftarrow T \cup \{e_i\}$<br>
            merge the sets containing $u$ and $v$<br>
        }<br>
    return $T$<br>
}

Kruskal



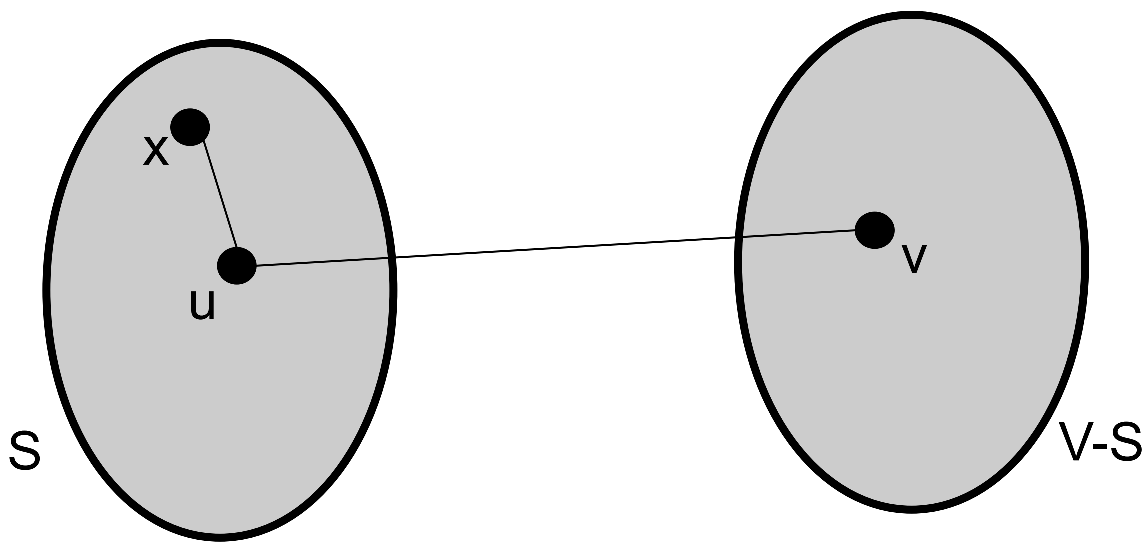
Sort edges weight.
Add edges whenever it does not create cycle.

# Cuts




In a graph $G = (V, E)$, a cut is a bipartition of V into disjoint sets $S, V - S$ for some $S \subseteq V$. We denote it by $(S, V - S)$.

An edge $e = \{u, v\}$ is in the cut $(S, V - S)$ if exactly one of $u, v$ is in $S$.

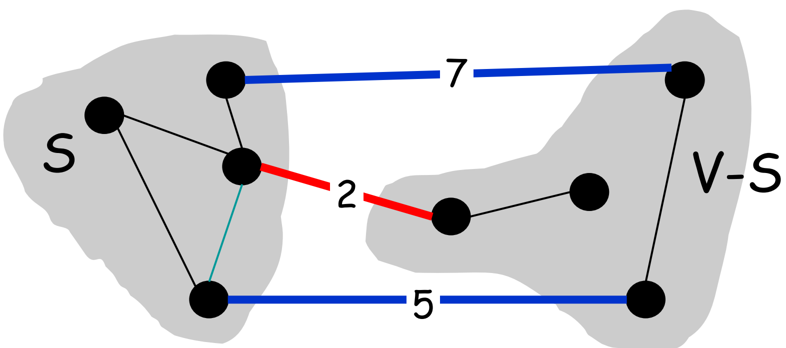


# Properties of the OPT

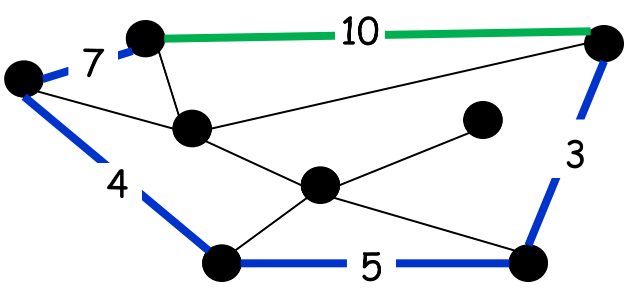
Simplifying assumption: All edge costs $c_e$ are distinct.

Cut property: Let $S$ be any subset of nodes (called a cut), and let $e$ be the min cost edge with exactly one endpoint in $S$. Then every MST contains $e$.

Cycle property. Let $C$ be any cycle, and let $f$ be the max cost edge belonging to $C$. Then no MST contains $f$.




red edge is in the MST




Green edge is not in the MST

# Cut Property: Proof

Simplifying assumption: All edge costs $c_e$ are distinct.

Cut property. Let $S$ be any subset of nodes, and let $e$ be the min cost edge with exactly one endpoint in $S$. Then any MST $T^*$ contains $e$.

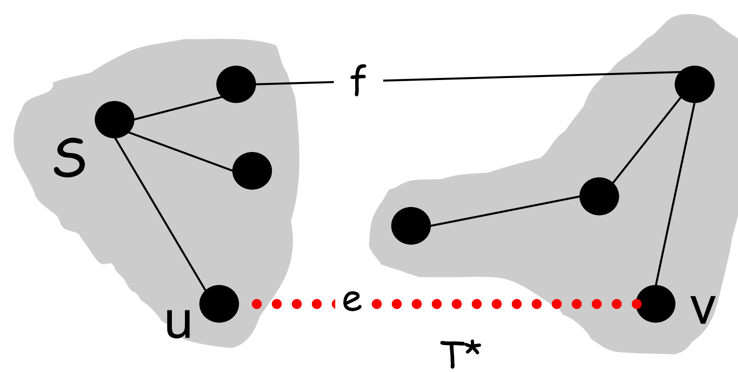
Proof. By contradiction

Suppose $e = \{u, v\}$ does not belong to $T^*$.

There is a path from $u$ to $v$ in $T^*$ $\Rightarrow$ there exists another edge, say $f$, that leaves $S$.

Adding $e$ to $T^*$ creates a cycle $C$ in $T^*$. (coz all tree has $n - 1$ edges)

> $T = T^* \cup \{e\} - \{f\}$ is also a spanning tree.
>
> Since $c_e < c_f$, $c(T) < c(T^*)$.
>
> This is a contradiction.



# Cycle Property: Proof

Simplifying assumption: All edge costs $c_e$ are distinct.

Cycle property: Let $C$ be any cycle in $G$, and let $f$ be the max cost edge belonging to $C$. Then the MST $T^*$ does not contain $f$.

Proof. By contradiction

Every connected graph has a spanning tree.
Hence it has at least $n - 1$ edges.

Suppose $f$ belongs to $T^*$.

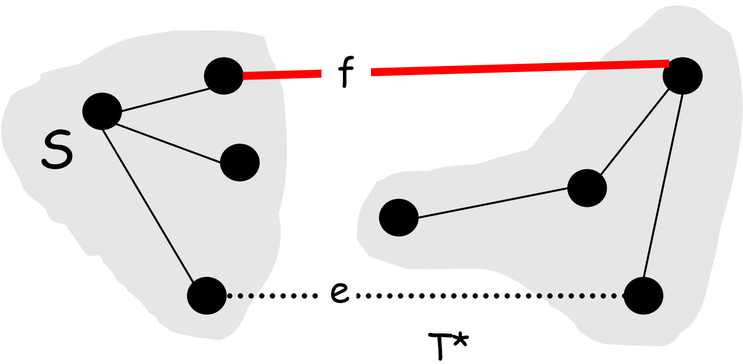
Deleting $f$ from $T^*$ cuts $T^*$ into two connected components.

There exists another edge, say $e$, that is in the cycle and connects the components.

$T = T^* \cup \{e\} - \{f\}$ is also a spanning tree.

Since $c_e < c_f$, $c(T) < c(T^*)$.

This is a contradiction.



# Proof of Correctness (Kruskal)

Consider edges in ascending order of weight.

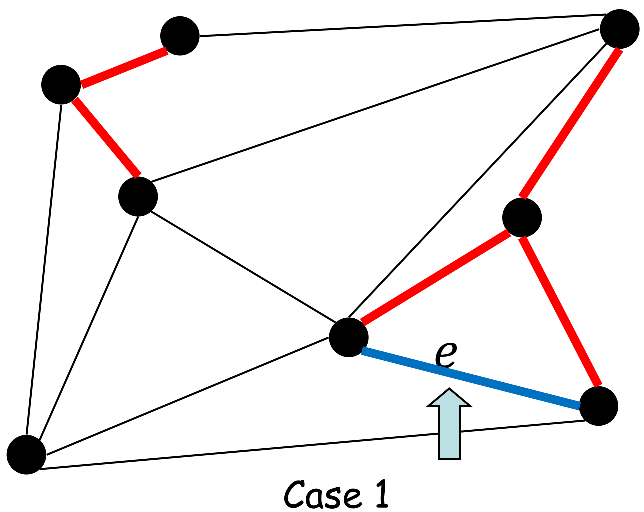
Case 1: adding $e$ to $T$ creates a cycle,

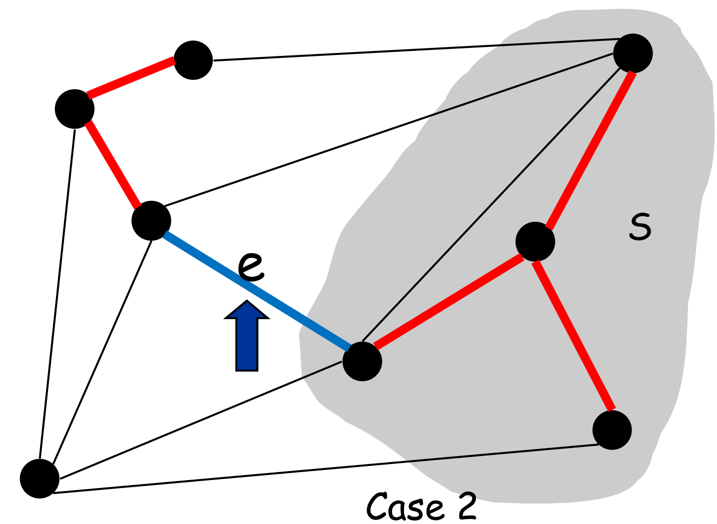
$e$ is the maximum weight edge in that cycle.

cycle property show $e$ is not in any minimum spanning tree.

Case 2: $e = (u, v)$ is the minimum weight edge in the cut $S$ where $S$ is the set of nodes in $u$'s connected component.

So, $e$ is in all minimum spanning tree.




This proves MST is unique if weights are distinct.

# Summary

- Greedy algorithm: ‘Best’ current partial solution at each step

- Design greedy algorithm:
  - How to order your input
  - Strategy for every step

- Greedy Analysis Strategies
  - Greedy algorithm stays ahead
  - Structural
  - Exchange argument