CS 401

Closest Points / Integer Multiplication / Matrix Multiplication

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Finding the Closest Pair of Points

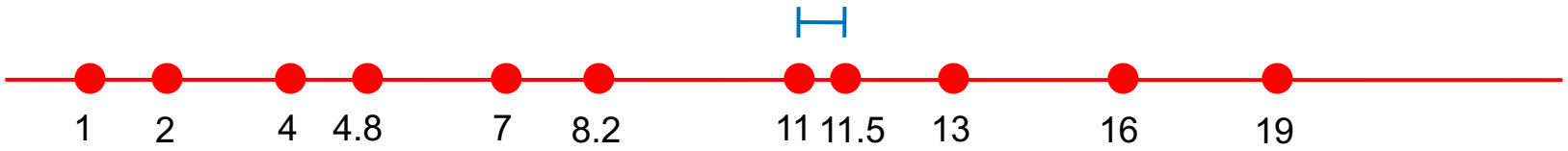
Closest Pair of Points

Given n points, find the closest pair.

Brute force: Check all $\binom{n}{2}$ pairwise distances

1-dim case: 11, 2, 4, 19, 4.8, 7, 8.2, 16, 11.5, 13, 1

find the closest pair



Fact: Closest pair is adjacent in ordered list

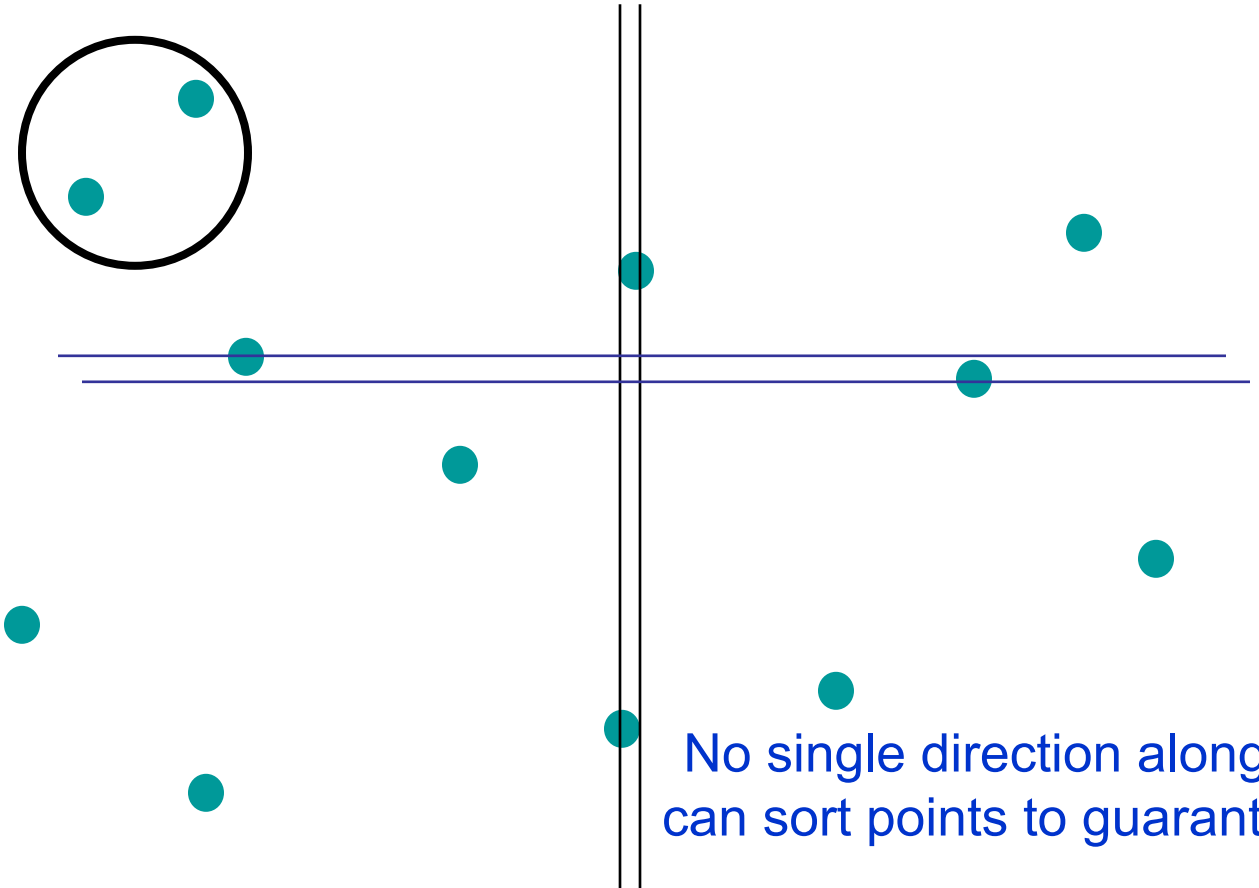
So, first sort, then scan adjacent pairs.

Time $O(n \log n)$ to sort, if needed, Plus $O(n)$ to scan adjacent pairs

Closest Pair of Points (2-dim)

Given n points in the plane, find a pair with smallest Euclidean distance between them.

Idea: make use of 1-dim algorithm (but not in a simple way)



No single direction along which one can sort points to guarantee success!

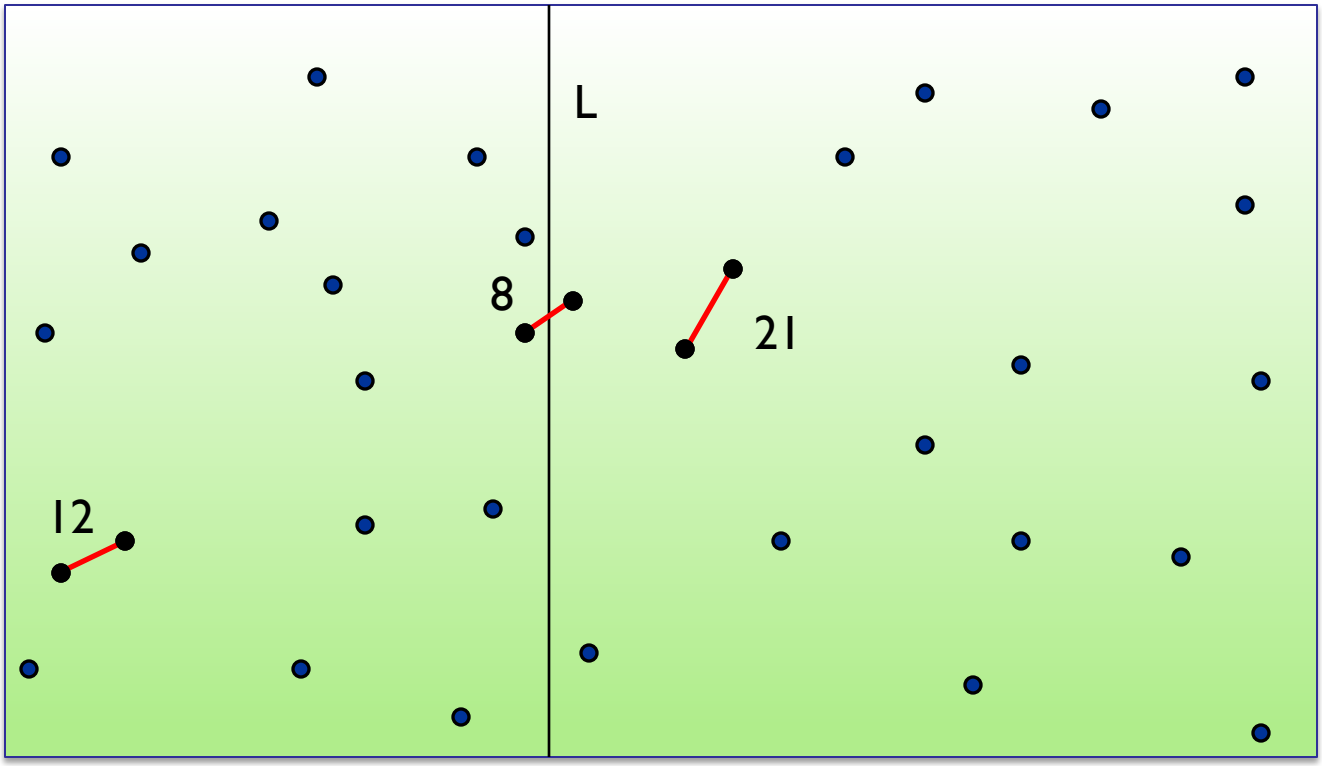
Divide & Conquer

Divide: draw vertical line L with $\approx n/2$ points on each side.

Conquer: find closest pair on each side, recursively.

Combine to find closest pair overall ← How ?

Return best solutions



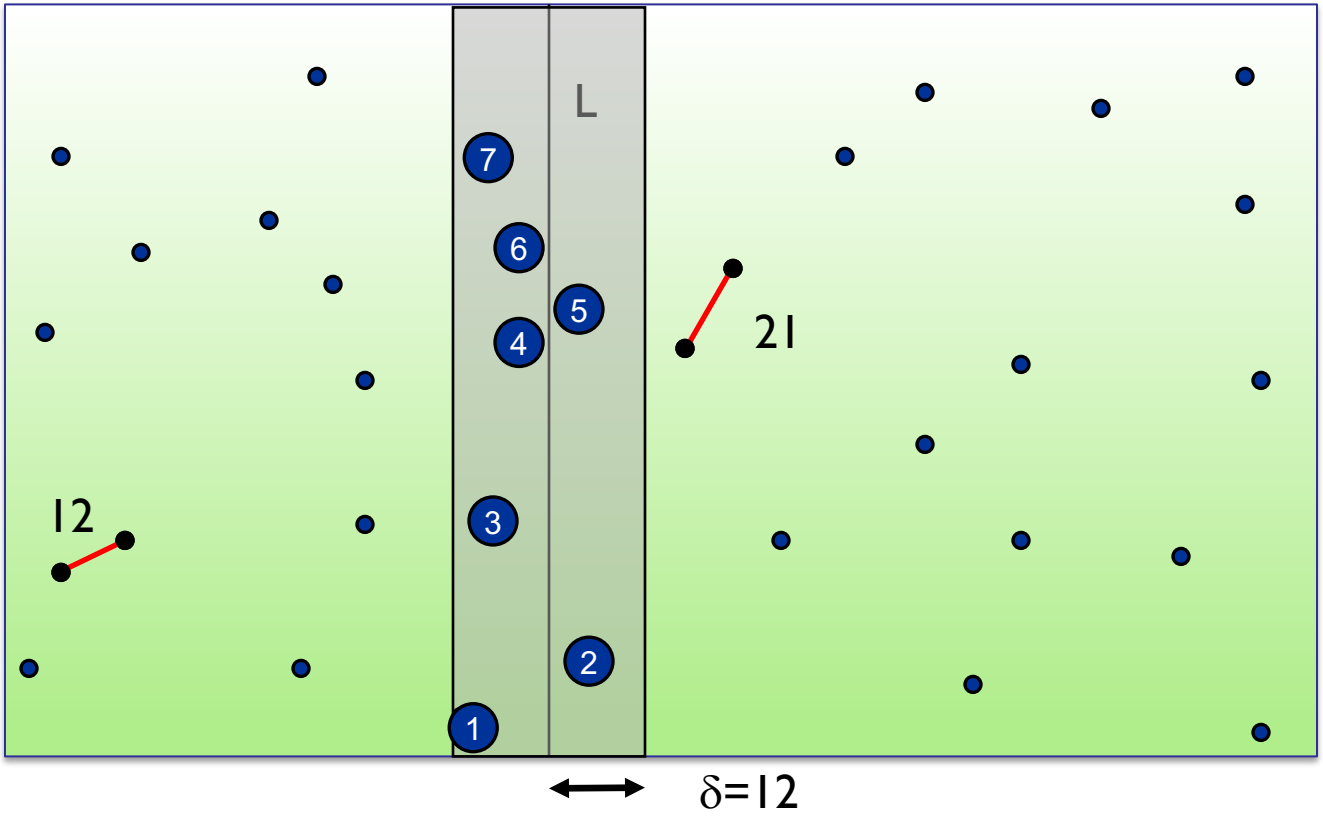
Key Observation

Suppose δ is the minimum distance of all pairs in left/right of L.

$$\delta = \min(12,21) = 12.$$

Key Observation: suffices to consider points within δ of line L.

Almost the one-D problem again: Sort points in 2δ-strip by their y coordinate.



Almost 1D Problem

Partition each side of L into $\frac{\delta}{2} \times \frac{\delta}{2}$ squares

Claim: No two points lie in the same $\frac{\delta}{2} \times \frac{\delta}{2}$ box.

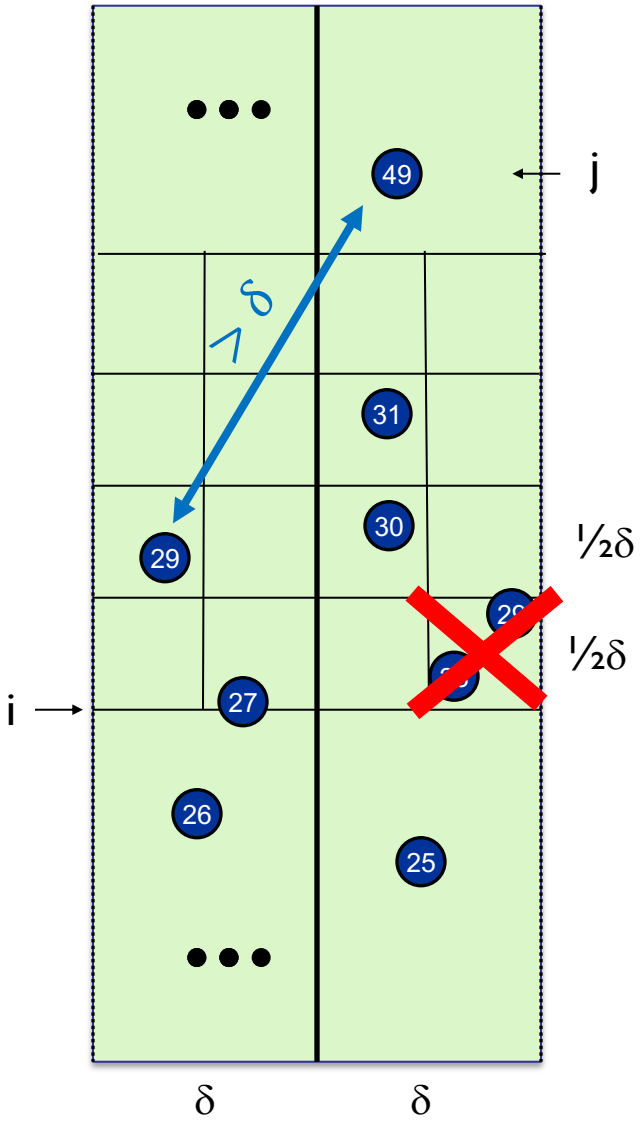
Proof: Such points would be within

$$\sqrt{\left(\frac{\delta}{2}\right)^2 + \left(\frac{\delta}{2}\right)^2} = \delta\sqrt{\frac{1}{2}} \approx 0.7\delta < \delta$$

Let s_i have the i^{th} smallest y-coordinate among points in the 2δ-width-strip.

Claim: If $|i - j| > 11$, then the distance between s_i and s_j is $> \delta$.

Proof: only 11 boxes within δ of $y(s_i)$.



Closest Pair (2 dimension)

```
Closest-Pair(p_1, p_2, ..., p_n) {
    if(n ≤ 2) return distance(p_1, p_2)

    Compute separation line L such that half the points
    are on one side and half on the other side.                    O(n log n)

    δ_1 = Closest-Pair(left half)
    δ_2 = Closest-Pair(right half)
    δ   = min(δ_1, δ_2)                                            O(1)

    Delete all points further than δ from separation line L        O(n)

    Sort remaining points p[1]…p[m] by y-coordinate.               O(n log n)

    for i = 1, 2, ..., m
        for k = 1, 2, ..., 11
          if i + k ≤ m                                             O(n)
              δ = min(δ, distance(p[i], p[i+k]));

    return δ.
}
```

Closest Pair Analysis

Running time?

$$T(n) \leq \begin{cases} 1 & \text{if } n \leq 2 \\ 2T\left(\frac{n}{2}\right) + O(n \log n) & \text{o. w.} \end{cases} \Rightarrow T(n) = O(n \log^2 n)$$

Can we do better?

Closest Pair (2 dimension) Improved

```
Closest-Pair(p_1, p_2, ..., p_n) {
   if(n ≤ 2) return distance(p_1, p_2)

   Compute separation line L such that half the points
   are on one side and half on the other side.                    O(1)

  (δ_1, Q_1) = Closest-Pair(left half)
  (δ_2, Q_2) = Closest-Pair(right half)
   δ         = min(δ_1, δ_2)                                     O(1)
   Q_sorted  = merge(Q_1, Q_2)  (merge sort it by y-coordinate)   O(n)

   Let S be points (ordered as Q_sorted) that is δ from line L.  O(n)

   for i = 1, 2, ..., m
      for k = 1, 2, ..., 11                                       O(n)
        if i + k ≤ m
            δ = min(δ, distance(S[i], S[i+k]));

   return δ and Q_sorted.
}
```

Assume: input sorted by x-coordinate
(O(n log n) overhead initially)

Input sorted by
y-coordinate

$$T(n) \leq \begin{cases} 1 & \text{if } n = 1 \\ 2T\left(\frac{n}{2}\right) + O(n) & \text{o.w.} \end{cases}$$

$$\Rightarrow T(n) = O(n \log n)$$

Summary

Closest pair in 2-dimension: Given n points in the plane, find a pair with smallest Euclidean distance between them.

Brute Force: Check all pairs of points in $\Theta(n^2)$ time.

Divide and Conquer:

- Divide: draw vertical line L with $\approx n/2$ points on each side.
- Conquer: find closest pair on each side, recursively.
- Combine to find closest pair overall

Exercise: Remove the assumption of "no two points have same x coordinate"?

Integer Multiplication

Integer Arithmetic

Add: Given two n-bit integers a and b, compute $a + b$.

Add

	1	1	1	1	1	1	0	1
		1	1	0	1	0	1	0	1
+	0	1	1	1	1	1	0	1
1	0	1	0	1	0	0	1	0

$O(n)$ bit operations.

Multiply: Given two n-bit integers a and b, compute $a \times b$. The "grade school" method:

Multiply

					1	1	0	1	0	1	0	1			
				*	0	1	1	1	1	1	0	1			
					1	1	0	1	0	1	0	1			
				0	0	0	0	0	0	0	0				
			1	1	0	1	0	1	0	1					
		1	1	0	1	0	1	0	1						
	1	1	0	1	0	1	0	1							
1	1	0	1	0	1	0	1								
	1	1	0	1	0	1	0	1							
		1	1	0	1	0	1	0	1						
			0	0	0	0	0	0	0	0					
0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	1

$O(n^2)$ bit operations.

Divide and Conquer

Let x, y be two n-bit integers

Write $x = 2^{n/2}x_1 + x_0$ and $y = 2^{n/2}y_1 + y_0$

where x_0, x_1, y_0, y_1 are all $n/2$-bit integers.

$$x = 2^{n/2} \cdot x_1 + x_0$$
$$y = 2^{n/2} \cdot y_1 + y_0$$
$$xy = \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right)$$
$$= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot (x_1 y_0 + x_0 y_1) + x_0 y_0$$

Therefore,

$$T(n) = 4T\left(\frac{n}{2}\right) + \Theta(n)$$

We only need 3 values $x_1y_1, x_0y_0, x_1y_0 + x_0y_1$
Can we find all 3 by only 3 multiplication?

So,

$$T(n) = \Theta(n^2).$$

Key Trick: 4 multiplies at the price of 3

$$x = 2^{n/2} \cdot x_1 + x_0$$
$$y = 2^{n/2} \cdot y_1 + y_0$$
$$xy = \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right)$$
$$= 2^n \cdot x_1 y_1 + 2^{n/2} \cdot (x_1 y_0 + x_0 y_1) + x_0 y_0$$

$$\alpha = x_1 + x_0$$
$$\beta = y_1 + y_0$$
$$\alpha\beta = (x_1 + x_0)(y_1 + y_0)$$
$$= x_1 y_1 + (x_1 y_0 + x_0 y_1) + x_0 y_0$$
$$(x_1 y_0 + x_0 y_1) = \alpha\beta - x_1 y_1 - x_0 y_0$$

Key Trick: 4 multiplies at the price of 3

Idea

$$x = 2^{n/2} \cdot x_1 + x_0 \Rightarrow \alpha = x_1 + x_0$$
$$y = 2^{n/2} \cdot y_1 + y_0 \Rightarrow \beta = y_1 + y_0$$
$$xy = \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right)$$
$$= \underset{\text{A}}{2^n \cdot x_1 y_1} + \underset{\alpha\beta - A - B}{2^{n/2} \cdot (x_1 y_0 + x_0 y_1)} + \underset{\text{B}}{x_0 y_0}$$

To multiply two n-bit integers:

- Add two $n/2$ bit integers.
- Multiply three $n/2$-bit integers.
- Add, subtract, and shift $n/2$-bit integers to obtain result.

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n)$$

Key Trick: 4 multiplies at the price of 3

Theorem [Karatsuba-Ofman, 1962] Can multiply two n-digit integers in $O(n^{1.585\ldots})$ bit operations.

$$x = 2^{n/2} \cdot x_1 + x_0 \Rightarrow \alpha = x_1 + x_0$$
$$y = 2^{n/2} \cdot y_1 + y_0 \Rightarrow \beta = y_1 + y_0$$
$$xy = \left(2^{n/2} \cdot x_1 + x_0\right)\left(2^{n/2} \cdot y_1 + y_0\right)$$
$$= \underbrace{2^n \cdot x_1 y_1}_{A} + \underbrace{2^{n/2} \cdot (x_1 y_0 + x_0 y_1)}_{\alpha\beta - A - B} + \underbrace{x_0 y_0}_{B}$$

Integer Multiplication (Summary)

- Amusing exercise: generalize Karatsuba to do 5 size $n/3$ subproblems

  This gives $\Theta(n^{1.46\ldots})$ time algorithm

Date	Authors	Time complexity
<3000 BC	Unknown	$O(n^2)$
1962	Karatsuba	$O(n^{\log 3/\log 2})$
1963	Toom	$O(n\,2^{5\sqrt{\log n/\log 2}})$
1966	Schönhage	$O(n\,2^{\sqrt{2\log n/\log 2}}(\log n)^{3/2})$
1969	Knuth	$O(n\,2^{\sqrt{2\log n/\log 2}}\log n)$
1971	Schönhage–Strassen	$O(n\log n\log\log n)$
2007	Fürer	$O\left(n\log n\,2^{O(\log^* n)}\right)$
2014	Harvey-Hoeven-Lecerf	$O(n\log n\,8^{\log^* n})$

Still open problem.

Matrix Multiplication

Multiplying Matrices

Let A be an $n \times m$ matrix, B be an $m \times p$ matrix.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{pmatrix}$$

Then, $C = AB$ is an $n \times p$ matrix

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{np} \end{pmatrix}$$

such that

$$c_{ij} = a_{i1}b_{1j} + \cdots + a_{im}b_{mj} = \sum_{k=1}^{m} a_{ik}b_{kj},$$

Multiplying Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} & \circ & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b_{44} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} & \circ & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34}+a_{24}b_{44} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}+a_{34}b_{41} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42} & \circ & a_{31}b_{14}+a_{32}b_{24}+a_{33}b_{34}+a_{34}b_{44} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31}+a_{44}b_{41} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32}+a_{44}b_{42} & \circ & a_{41}b_{14}+a_{42}b_{24}+a_{43}b_{34}+a_{44}b_{44} \end{bmatrix}$$

Multiplying Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} & \circ & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b_{44} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} & \circ & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34}+a_{24}b_{44} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}+a_{34}b_{41} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42} & \circ & a_{31}b_{14}+a_{32}b_{24}+a_{33}b_{34}+a_{34}b_{44} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31}+a_{44}b_{41} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32}+a_{44}b_{42} & \circ & a_{41}b_{14}+a_{42}b_{24}+a_{43}b_{34}+a_{44}b_{44} \end{bmatrix}$$

Multiplying Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{matrix} \mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{A_{21}} & \mathbf{A_{22}} \end{matrix} \bullet \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} \begin{matrix} \mathbf{B_{11}} & \mathbf{B_{12}} \\ \mathbf{B_{21}} & \mathbf{B_{22}} \end{matrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}+a_{14}b_{42} & \circ & a_{11}b_{14}+a_{12}b_{24}+a_{13}b_{34}+a_{14}b_{44} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}+a_{24}b_{41} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42} & \circ & a_{21}b_{14}+a_{22}b_{24}+a_{23}b_{34}+a_{24}b_{44} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31}+a_{34}b_{41} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32}+a_{34}b_{42} & \circ & a_{31}b_{14}+a_{32}b_{24}+a_{33}b_{34}+a_{34}b_{44} \\ a_{41}b_{11}+a_{42}b_{21}+a_{43}b_{31}+a_{44}b_{41} & a_{41}b_{12}+a_{42}b_{22}+a_{43}b_{32}+a_{44}b_{42} & \circ & a_{41}b_{14}+a_{42}b_{24}+a_{43}b_{34}+a_{44}b_{44} \end{bmatrix}$$

$$\begin{matrix} \mathbf{A_{11}B_{11}+A_{12}B_{21}} & \mathbf{A_{11}B_{12}+A_{12}B_{22}} \\ \mathbf{A_{21}B_{11}+A_{22}B_{21}} & \mathbf{A_{21}B_{12}+A_{22}B_{22}} \end{matrix}$$

Simple Divide and Conquer

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$= \begin{pmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{pmatrix}$$

Strassen's Divide and Conquer Algorithm

Naive

$$\mathbf{C}_{1,1} = \mathbf{A}_{1,1}\mathbf{B}_{1,1} + \mathbf{A}_{1,2}\mathbf{B}_{2,1}$$
$$\mathbf{C}_{1,2} = \mathbf{A}_{1,1}\mathbf{B}_{1,2} + \mathbf{A}_{1,2}\mathbf{B}_{2,2}$$
$$\mathbf{C}_{2,1} = \mathbf{A}_{2,1}\mathbf{B}_{1,1} + \mathbf{A}_{2,2}\mathbf{B}_{2,1}$$
$$\mathbf{C}_{2,2} = \mathbf{A}_{2,1}\mathbf{B}_{1,2} + \mathbf{A}_{2,2}\mathbf{B}_{2,2}$$

Strassen

$$\mathbf{M}_1 := (\mathbf{A}_{1,1} + \mathbf{A}_{2,2})(\mathbf{B}_{1,1} + \mathbf{B}_{2,2})$$
$$\mathbf{M}_2 := (\mathbf{A}_{2,1} + \mathbf{A}_{2,2})\mathbf{B}_{1,1}$$
$$\mathbf{M}_3 := \mathbf{A}_{1,1}(\mathbf{B}_{1,2} - \mathbf{B}_{2,2})$$
$$\mathbf{M}_4 := \mathbf{A}_{2,2}(\mathbf{B}_{2,1} - \mathbf{B}_{1,1})$$
$$\mathbf{M}_5 := (\mathbf{A}_{1,1} + \mathbf{A}_{1,2})\mathbf{B}_{2,2}$$
$$\mathbf{M}_6 := (\mathbf{A}_{2,1} - \mathbf{A}_{1,1})(\mathbf{B}_{1,1} + \mathbf{B}_{1,2})$$
$$\mathbf{M}_7 := (\mathbf{A}_{1,2} - \mathbf{A}_{2,2})(\mathbf{B}_{2,1} + \mathbf{B}_{2,2})$$

$$\mathbf{C}_{1,1} = \mathbf{M}_1 + \mathbf{M}_4 - \mathbf{M}_5 + \mathbf{M}_7$$
$$\mathbf{C}_{1,2} = \mathbf{M}_3 + \mathbf{M}_5$$
$$\mathbf{C}_{2,1} = \mathbf{M}_2 + \mathbf{M}_4$$
$$\mathbf{C}_{2,2} = \mathbf{M}_1 - \mathbf{M}_2 + \mathbf{M}_3 + \mathbf{M}_6$$

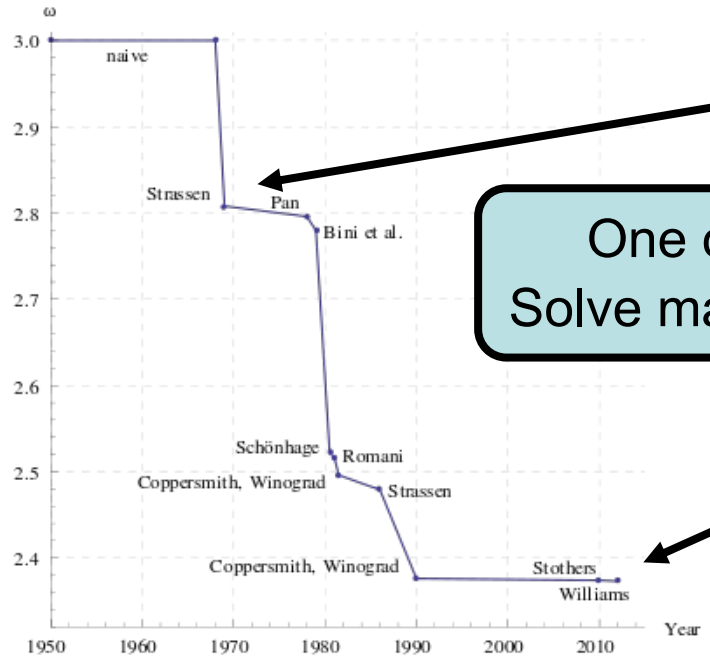
Strassen's Divide and Conquer Algorithm

- Strassen's algorithm

  Multiply 2×2 matrices using 7 instead of 8 multiplications (and 18 additions)

$$T(n) = 7T\left(\frac{n}{2}\right) + 18n^2$$

  Hence, we have $T(n) = O\left(n^{\log_2 7}\right)$.



Useful when $n\sim500$.

One of the most important open problem:
Solve matrix multiplication in $O(n^2\log^{O(1)}n)$ time

I am curious how large n need?