Homework 1: Due May 9, 2022

Please answer at least **three** of the following questions.

Problem 1:
Recall that in the randomized dynamic connectivity algorithm, for each value of $1/2, 1/4, 1/8, \ldots,$ we sample graphs by sampling each edge with the given value. In addition, for each sampled graph, we assign each edge a random binary string as the name for the edge so that we can use exclusive or to sample edges going out from a set of vertices.
Assume the graph has n vertices, and we need to handle $t = O(n)$ graph updates. Assume for each probability, we maintain $O((\log n)^4)$ sampled graphs. How many bits of the binary names for edges are needed to make sure that all the updates are handled correctly with probability $1 - 1/n$? Briefly give your justification.

Problem 2:
Present an algorithm to construct a spanning tree for a given unweighted graph $G = (V, E)$ with total stretch $O(m^{1+o(1)})$ (all the edges used must be in E). In particular, specify the parameters you used. No justification.

Problem 3:
Prove that in the Bartal's algorithm to construct low stretch tree, $\mathbb{E}[d_T(u,v)] \le \frac{d_G(u,v)\cdot \log n}{\Delta/2}$, where Δ is the diameter of the graph, $d_T(u,v)$ is the distance between two vertices in the Bartal tree, and $d_G(u,v)$ is the distance between two vertices in the given graph.

Problem 4:
Let L_G be the laplacian matrix of a graph. Prove the following statements:

1. 0 is always an eigenvalue of L_G.
2. The dimension of eigenvectors for eigenvalue 0 is the same as the number of connected components in G.

Problem 5:
Given a graph that is a ϕ-expander. Present an algorithm to find all the cuts of size at most c in time $O(n \cdot (c/\phi)^{O(c)})$. No justification.