CS 594: Representations in Algorithm Design **Spring 2022**

Lecture 7: 02/15/2022

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1 Last Lecture:

- Courant-Fischer and Cheeger's Inequality
- Expanders
- Introduced to Expander Decomposition

2 Expanders:

Given a graph G that is a Φ expander, when G has a small cut size c then one of the sides of the cut will be smaller than the other. Cut the graph such that one side is smaller than other. Then the smaller side will have a lesser running time. The expanders equation is given as follows :

$$\min \text{ V} \supsetneq S \frac{|E(S,\overline{\text{S}})|}{minVo|(\overline{\text{S}}),Vo|(S)} \geq \Phi$$

3 Expander Decomposition:

P is a (Φ, ε) expander decomposition where P is a vertex Partition for the given Graph G if Q ∈ P, G(Q) is a Φ expander where Φ is given with respect to the graph (Might equal to $\frac{1}{logn}$ and number of edges crossing expander is at most $\varepsilon \cdot$ m . m here is the number of edges of the graph.

More formally, given a graph G = (V, E) we aim to find a partitioning of V into V1, . . . , Vk for some k, such that the total number of edges across different clusters is small while the conductance of each cluster as an induced subgraph is large. This bicriteria measure is advantageous over other popular measures such as min diameter decomposition, k-center, and k-median since there are simple examples where these measures fail to capture the natural clustering. Moreover, expander decomposition has seen great applications in algorithm design including graph sketching/sparsification, undirected/directed Laplacian solvers, max flow algorithms , approximation algorithms for unique game, and dynamic minimum spanning forest algorithms

Question : If exist a $0 < \Phi < 1$ then how small can the ϵ be?

Answer : It means there exist a graph G, whose number of crossing edges is always greater than Log n.

4 Leighton-Rao 1999

There is a polynomial time 0(m2) algorithm to find a S in a given Graph G, such that

$$\frac{|E(S,\overline{\text{S}})|}{minVo|(\overline{\text{S}}),Vo|(S)} \leq \text{log min T} \subset V(\frac{|E(T,\overline{\text{T}})|}{minVo|(\overline{\text{T}}),Vo|(T)})$$

Leighton-Rao algorithm uses LP Relaxation and Rounding.

Question : Given a Graph G, Certify G is a Φ exapnder or Find a Φ sparse cut that recurse on both sides of cut .

Answer : We have to run Leighton-rao alogrithm or Expander Decomposition . There must be a partition of the graph cut such that , $\epsilon \leq (\phi \text{ log n})$

$$\frac{|E(S,\overline{\text{S}})|}{minVo|(\overline{\text{S}}),Vo|(S)} \leq \Phi$$

Lemma : It is a NP hard to find S that minimize ,

$$\frac{|E(S,\overline{\mathrm{S}})|}{minVo|(\overline{\mathrm{S}}),Vo|(S)}$$

$$\text{min V} \supsetneq S\frac{|E(S,\overline{\mathrm{S}})|}{minVo|(\overline{\mathrm{S}}),Vo|(S)} \geq \Phi$$

5 Expander Decomposition[Kamma-Vempula-Vetta 2000]

The expander decomposition in O(m^3)

Dynamic Connectivity can be done in $\mathbf{O(1)}$ time . We need to use hash to sample. Sometimes the sample might fail. If we do in deterministic way, we need to combine the idea of making use of expander property .

OBSERVATION: Given a partition of a ϕ expander and a partition cut of expander (S, $\overline{S}$) we can certify that S and $\overline{S}$ are not connected or there must be some other edge $\overline{E}$ (S, $\overline{S}$ in polynomial time (1/ ϕ) .

$$minVo|(\overline{\mathrm{S}}),Vo|(S) \leq c/\Phi$$

Algorithm : Suppose if edge E, O/ (U,V $\in$ S, $\overline{S}$) of Graph G is deleted , Find all cuts of size $\leq$ C/ ϕ which contacts X in time $\mathbf{O(C/}\phi\ )^C$

Answer : Do Breadth First Search algorithm from vertex U and If it have a small cut C , one of the both sides should be small.

6 Expander Pruning [Saranurak-Wang 2019]

Given a graph G that is a Φ expander,delete K edges from G and k $\leq \Phi$m/10 there is an algorithm to compute T such that the Vo(T) $\leq \delta$k/ϕ to find S $\subseteq$ V S, t. The three conditions are :

1. Vol(S) $\leq$ 8k/ϕ
2. E(S,S) $\leq$ 4k
3. G(S)is $\phi/6$ expander after removing the edges

The total time for updating is O(k log $\mathbf{m}/\phi^2$).

The rest is convered in the next class .