CS 594: Representations in Algorithm Design **Spring 2022**

Lecture 13: Feb 22, 2022

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1 Last time and today

Previously: Graph decomposition (low-diameter, expander decomposition)

- Utilize nice properties on good edges (intra-component)
- Upper bound number of bad edges (inter-component)

Starting today: Graph sparsification (edge, vertex sparsification)

- Preserve some properties of the original graph
- Bound the size of sparsified graph for computation efficiency

2 Graph sparsification

2.1 Edge sparsification

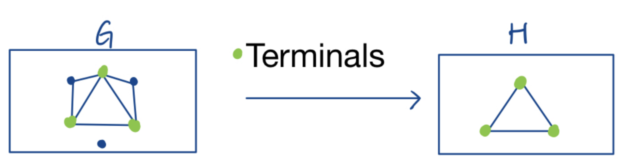
$$G = (V, E) \longrightarrow H = (V, E')$$

Such that: property on $H \approx$ property on G, and $|E'| \ll |E|$.
Eg., Spanning forest for graph connectivity property.

2.2 Vertex sparsification

$$G = (V, E),\ \text{(terminal set)}\ T \subset V \longrightarrow H = (V', E')$$

Such that: $T \subset V'$, property on T in $H \approx$ property on T in G, $|V'| \ll |V|$, $|E'| \ll |E|$.
Eg., weighted graph for distance property.



3 Vertex sparsification for min-cut property

3.1 The problem

Given $G = (V, E)$, $T \subset V$, construct a vertex sparsifier $H = (V', E')$ such that: $T \subset V'$, and for all partitions of T into $(T', T \setminus T')$:

$$\text{mincut}_H(T', T \setminus T') = \text{mincut}_G(T', T \setminus T').$$

With additional parameter of threshold c:

If $\text{mincut}_G(T', T \setminus T') \leq c$, then $\text{mincut}_H(T', T \setminus T') = \text{mincut}_G(T', T \setminus T')$.
If $\text{mincut}_G(T', T \setminus T') > c$, then $\text{mincut}_H(T', T \setminus T') > c$.

3.2 The algorithm

Definition 1 (Cut containment set) *A set of edges E' in G is called a cut containment set if $\forall (T', T \setminus T')$ with $\text{mincut}_G(T', T \setminus T') \leq c, \exists$ such a min-cut with edges C and $C \subset E'$.*

Definition 2 (Intersecting-all-mincut set (old definition)) *Given $G = (V, E)$, terminal set $T \subset V$ and threshold c, we say a set of edges F in G is a set of edges intersecting all min terminal cuts if $\forall (T', T \setminus T')$ with $\text{mincut}_G(T', T \setminus T') \leq c, \exists$ such a min-cut with edges C and $C \cap F \neq \emptyset$.*

Definition 3 (Intersecting-all-mincut set (new definition)) *Given $G = (V, E)$, terminal set $T \subset V$ and threshold c, we say a set of edges F in G is a set of edges intersecting all min terminal cuts if after removing F from G, $\forall (T', T \setminus T')$ with $\text{mincut}_G(T', T \setminus T') \leq c$, no connected component in $G \setminus F$ contains all the edges of a terminal min-cut.*

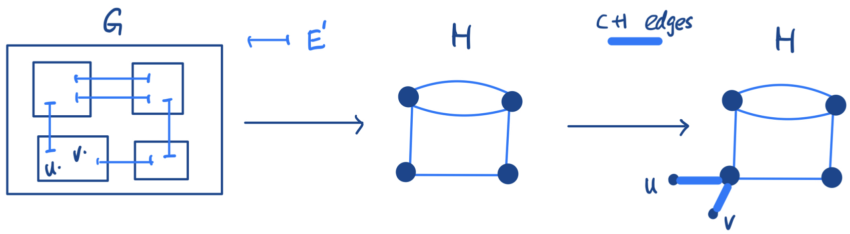
Theorem 4 *For vertex sparsification with respect to the min-cut property with threshold parameter c, there exists a sparsifier of size $|T| \cdot c^{O(c)}$. (Can be improved to $|T| \cdot c^3 \cdot \log^4 n$.)*

Suppose we have a cut containment set E', we can construct a vertex sparsifier H from E' in the following way:

- Remove edges in E' from G, breaking G into several connected components.

- Shrink each connected component into 1 super-vertex in H.
- Connect vertices in H with corresponding edges in E'.
- If terminal vertices u, v are merged into the same connected component (thus the terminal nodes are lost in H), add u, v as vertices in H, connect each with $c+1$ parallel edges to the super-vertex corresponding to the connected component they're in. (There were no min-cut of size $\leq c$ between them in G.)

Resulting H has number of edges (and vertices) $\leq |E'| + |T| \cdot (c+1)$. Next we need to upper-bound size of E'.



Claim 5 *Given G, T, c, $\exists E'$ st. $|E'| = |T| \cdot c^{O(c)}$.*

Proof idea: recursively build E' using intersecting-all-set F:

- Initialize $E' = \emptyset$.
- Find intersecting-all-set F_c in G for given $T = T_0$ and c.
- Remove all edges in F_c from G, add them to E', and add endpoints of edges in F_c to terminals:
  $G \leftarrow G \setminus F_c$, $E' \leftarrow E' \cup F_c$, $T_1 \leftarrow T_0 \cup \{u, v \mid (u, v) \in F_c\}$.
- Continue to find intersecting-all-set F_{c-1} with updated G and T_1, until $c = 1$.
- Return E' (which is $\cup_{i=1}^{c} F_c$).

Claim 6 *Given G, T, c, the intersecting-all-set F_c is of size $|F_c| \leq |T| \cdot c$. (Each terminal vertex is incident to at most c edges that are on any qualifying min terminal cut.)*

Proof of Claim 5 by induction:
Base case: $|F_c| \leq |T| \cdot c$.
Step 1: $|T_1| \leq |T| + 2|F_c| \leq (1 + 2c) \cdot |T| = O(c) \cdot |T|$, and $|F_{c-1}| \leq |T_1| \cdot c = O(c^2) \cdot |T|$.
General induction step: $|T_i| \leq |T_{i-1}| + 2|F_{c-i+1}| = O(c^i) \cdot |T|$, $|F_{c-i}| = O(c^{i+1}) \cdot |T|$.
Finally, $|F_1| = O(c^c) \cdot |T|$.
$|E'| = | \cup_{i=1}^{c} F_c| = c^{O(c)} \cdot |T|$.