CS 594: Representations in Algorithm Design	Spring 2022

Lecture on 03/01/2022

Lecturer: Xiaorui Sun *Scribe: Kushal Reddy Palvai*

1 Last Lecture's Review

In the last lecture, we discussed how the edge sparsification of the graph works.

2 Edge Sparsification of graph

Given a G = (V,E) -> H = (V,E') We need to show that graph property on H is approximately the same as on G.

|E'| « |E| we do Edge Sparsification versus (V/S) Vertex Sparsification, where Edge Sparsification is weaker compared to Vertex Sparsification which is comprehensively stronger.

3 Laplacian

$L_G \approx L_H$

G = (V,E,W)

deg(V) = $\sum_{(V,X) \in E}$ W(V,X)

1. L_G, This is a Symmetric Matrix. It is also a positive semi - definite matrix. $\forall X X^+ L_G X \geq 0$
2. L_G, Eigen values are real and non negative. 0 is always an Eigen value.

4 PSD Matrix

we take a PSD Matrix A,B

A $\approx_\epsilon$ B

$(1-\epsilon).B \preceq A \preceq (1+\epsilon).B$

where $\preceq$ is the Loewner ordering of PSD matrices

A $\preceq B <=> \forall X X^{+} A X \leq X^{+}.B.X$
Find H, Show that Goal: L $_{G} \approx_{\epsilon}$ L $_{H}$
$S_{(\mathrm{S},\overline{S})} \subseteq V$
X $_{S}$ (V) = $\{1\ V \in S$
X $_{S}$ (V) = $\{1\ V \in \overline{S}$
X $_{S}$ ^{T}. L $_{G}$.X $_{S}$ = 2 Cut Size (S,$\overline{S}$)
G,b,L$_{G}$.X = b $\leftarrow LaplacianSystem$

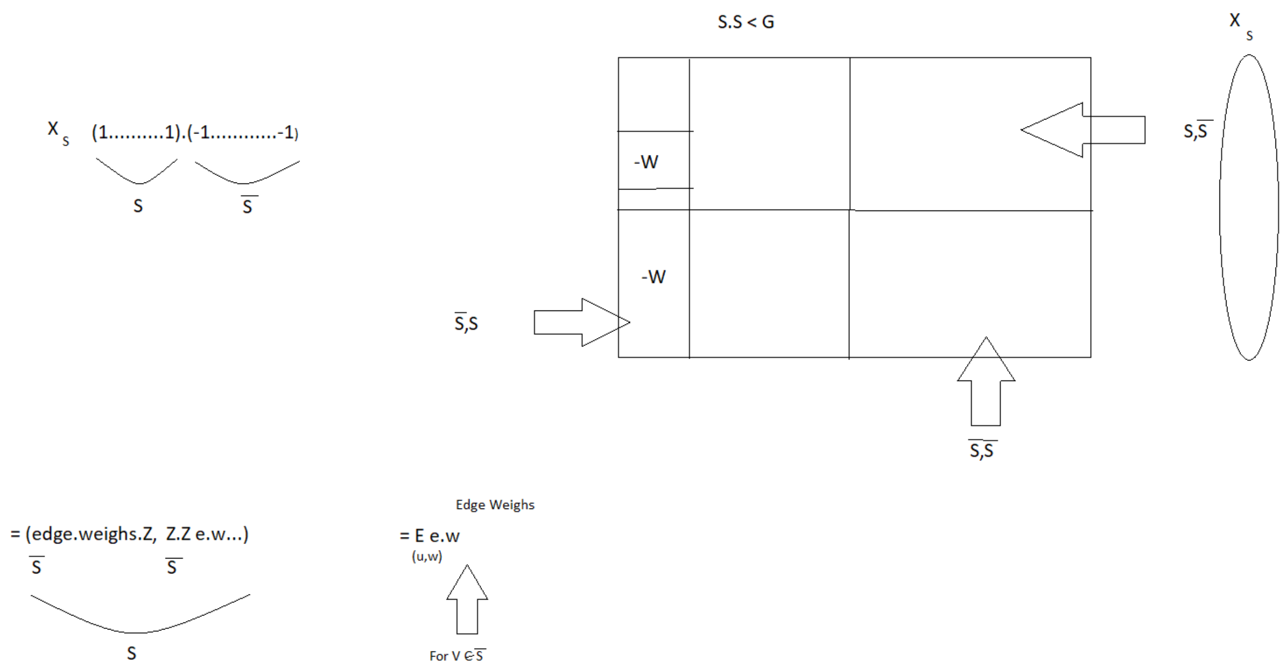

Figure 1: Implementation in PSD Matrix

X $_{S}$ = Z.$\overline{Z}_{(\mathrm{U,W})} Edgeweight$
X $_{S}$ = 4 $\overline{Z}_{(\mathrm{U,W})}.W(U,X)$

5 Solve Linear System

A x = b O(n^{3}) (where A $\leftarrow nxn$)

By Matrix Multiplication: $O(n^w)$
where w = Constant for base matrix multiplication algorithm
$2 \leq w \leq 3$
$w = 2.37....$

Solve Laplacian Matrix $O(m.polylog(n))$
Approximate Algorithm for graph maxflow in O(m).
Exact algorithm for maxflow $O(m^{3/2})$

6 Theorem: Algorithm to construct Edge Sparsification of L_G

$O(n.logn \div \epsilon^2) edges$
Algorithm: Independently, Sample edges with probability Depending on G.
For each edge, e ϵE
where, $P_e \leftarrow Sample\ Probability$
$\alpha_e \leftarrow Weight\ in\ Sparsification$

With Probability P_e, add e to H with α_e.
$- - - - - - - - - - - - - - - (1 - P_e), Discard\ e.$
$L_G = \epsilon_e L_e$
Y_e to be the Laplacian Contribution in sampling process for edge e.
$Y_e = \{0 matrix \ \ 1 - P_e$
$Y_e = \{\alpha_e L_e \ \ P_e$
$L_H = \epsilon_{e\ \epsilon E}\ Y_e \approx L_G$
$E[L_H] = L_G$
where E = Expectation, and Expectation of L_H is same as L_G.
$E[Y_e] = L_e$
Set $\alpha_e = 1/P_e$

7 Theorem: Matrix Concentration

$X_1, X_2, ...X_m$ is a Sequence of nxn Independent.
Random matrices show that
$0 \leq X_i \leq \epsilon^2/logn.I$

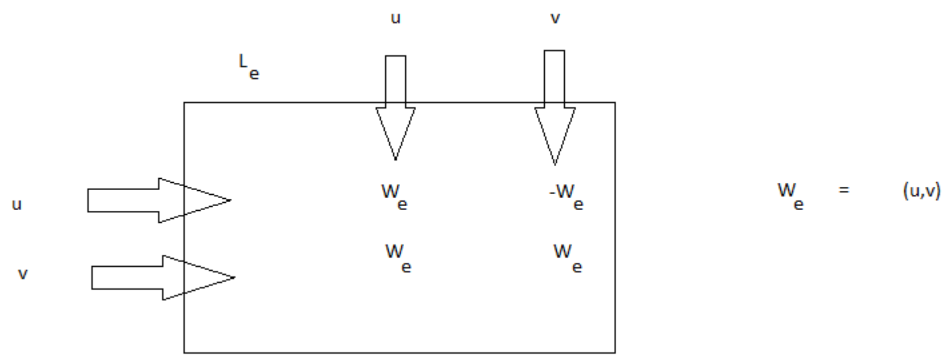

Figure 2: Ye to be the laplacian Contribution in sampling process for edge e

X $= \epsilon X_{\mathrm{i}}$
$if E[x] = I$
$\{x_{\mathrm{i}}, ...x_{\mathrm{n}}\}$
Then we have to prove X $\approx {}_{\mathrm{G}} I$
Fact: A $\leq B, \forall U,$
$U^{\mathrm{T}} A U \leq U^{\mathrm{T}}.B.U$

Want U

$U^T. L_H.U = I$
$U^T.L_G.U = I$
From This we get
$(L_G^{+/2})^T. L_G.(L_G^{+/2}) = I$
$V.V^T = I$
$X_e = (L_G^{+/2}). Y_e(L_G^{+/2})$
From this we can find the Expectation E, $E[\epsilon_e.X_e]$
$= (L_G^{+/2}).L_G.(L_G^{+/2})$

$\lambda_{max}(X_e) \leq \epsilon^2/logn$
$\lambda_{max}(1/P_e(L^{+/2}).L_e.(L^{+/2})) \leq \epsilon^2/logn$
$t_r[1/P_e(L^{+/2}).L_e.(L^{+/2})] \leq \epsilon^2/logn$
$P_e \geq (t_r[1/P_e(L^{+/2}).L_e.(L^{+/2})].Logn)/\epsilon^2$

———THE END———