CS 594: Representations in Algorithm Design **Spring 2022**

Lecture 7: 02/01/2022

Lecturer: Xiaorui Sun *Scribe: Shashwath Jawaharlal Sathyanarayan*

1 Last Lecture's Review:

In the previous few lectures, we were introduced to the concept that a tree T can represent a graph G(V,E), such that certain desirable properties about the graph are reflected within the representation. In particular, the concept of "Stretch" is discussed in detiled below.

- **(Low stretch spanning) tree:** Given and edge from the original graph eu,v E, the stretch of an unweighted edge is the distance between u and v in T : StretchT (e) = distT (u, v). We express the stretch of T across the entire graph is the sum of the stretch of all the edges. Goal: We want to produce a low stretch tree using randomness using some probability. T is the disribution on trees - T1, T2, T3, ..., Tk G is A-probability embedded to T if the below condition is met: $\mathcal{S}$ - :For each tree Ti $\in \mathcal{T}\rangle$, $d_T(u,v) \geq d_a(u,v)$ - $E_{T\in\mathcal{T}}\mathrm{d}_T(u,v) \geq d_a(u,v)$

  Relevant Readings:

- Lecture 6 scribe notes
- Review on the concept of a tree's stretch and how to generate a low stretch spanning tree
- Bartal's algorithm

2 Bartal's Theorem:

How low can we set the distortion factor so that, for any metric weighted graph G, we can find a distribution on trees D that allows an -probabilistic embedding of G? A theorem due to Bartal proposes such an and a way to construct D from any metric weighted graph G. Before proving the theorem, however, we introduce a procedure for finding a "Low Diameter Randomized Decomposition" (LDRD for short) of graphs. The randomized, recursive procedure LDRD takes as input a graph G = (V, E) and a parameter $\pi \geq 0$ and outputs a partition Vi - V such that:

- diam(G[Vi]) $\geq\sigma$ for all i, where diam($\cdot$) is the diameter function on graphs.
- Pr[edge i, j not in any E(G[Vk])] (d(i, j) log n)/ π. In other words, if the parts Vi were assigned distinct colors, this is an upper bound for the probability of an edge i, j being bichromatic.

Lemma: For each edge e= (u,v) P_r [e is an crossing edge] $\leq \frac{(d_a(v,u) \log n}{\Delta}$
Note: Distance on edge is large, then probability that edge e will bad is very high and vice versa
Observation: When e becomes an crossing edge ,
$\rightarrow C(u) \neq$ C(v)
Assume C(u)= i, C(v) = j

1. **Observation** If edge e is an crossing edge, $dist_a(\Delta(i), u) \leq R, dist_a(\Delta(i), v) > R$

Claim If edge e is cut by vertex x, then R $\in [dist_a(x, u), dist_a(x, 0)]$
Assumption: fix e=(u,v)
For another vertex x,
$L_x = min(dist_a(x, u), dist_a(u, v))$
$U_x = L_x + dist_a(u, v)$
The claim is equivalent to $R \in [L_x, U)$ when an edge is cut by X, then one necessary condition

1. If R < x, then u and v are both outside of ball of x, so x cannot assign values to u and v.
2. If R > x, then u and v are contained inside the ball of x so x can assign the colors and u ,v will be assigned same color by x.

Consider all the vertices that can potentially cut the edges here we take e = (u,v)
We consider L_x, U_x for the vertices and sort them according to L_x, U_x
1. For vertex X_j, L_{xj} and U_{xj}, if it overlaps in the interval$(\Delta/4, \Delta/2$, then vertex x_2 or x_n cannot cut edge e.
2. For X_j, L_{xj} and U_{xj}, if the $\Delta/4 > L_xj, U_xj < \Delta/2$,
what is necessary condition that x_j cut edge e for permutation π
Suppose we x_k and x_j where $k < j$
If Rank of $x_k <$ Rank of x_j , then x_j cannot cut edge e as x_k have smaller $L_{xk} < L_{xj}$

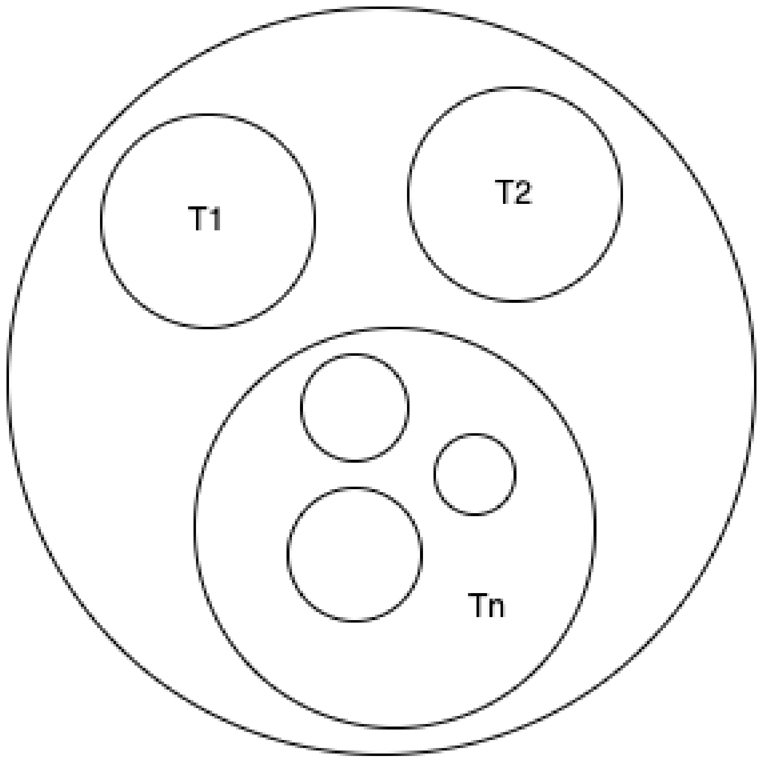

Figure 1:

that is x_k has potential of cutting edge e. and one of two vertices u,v , one of u,v $\in B(x_k, R)$
Once the x_k is processed,one of u,v will be colored most likely, both of the vertices will be colored by then, so x_j will not have a chance to color the vertices.

Necessary Condition: For X to cut edge e in permutation π, then $\pi(x_k) > \pi(x_j)$ for all k < j.
Observation: For x_j to cut edge e, following two conditions should be satisfied,

1. R $\in [L_x j, U_x j]$
2. k < j , $\pi^{-1}(x_k) > \pi^{-1}(x_j)$ i.e. Rank of $x_k >$ Rank of x_j

Based on 2 observations , find the upper bound of P_r[e is cut by x_j]
For a Random sample R in the range $(\Delta/4, \Delta/2)$, the probability that R falls in between $[L_{xj}, U_{xj}]$ is given by $P_r[R \in [L_{xj}, U_{xj}]] \leq \frac{d(u,v)}{\Delta/4}$

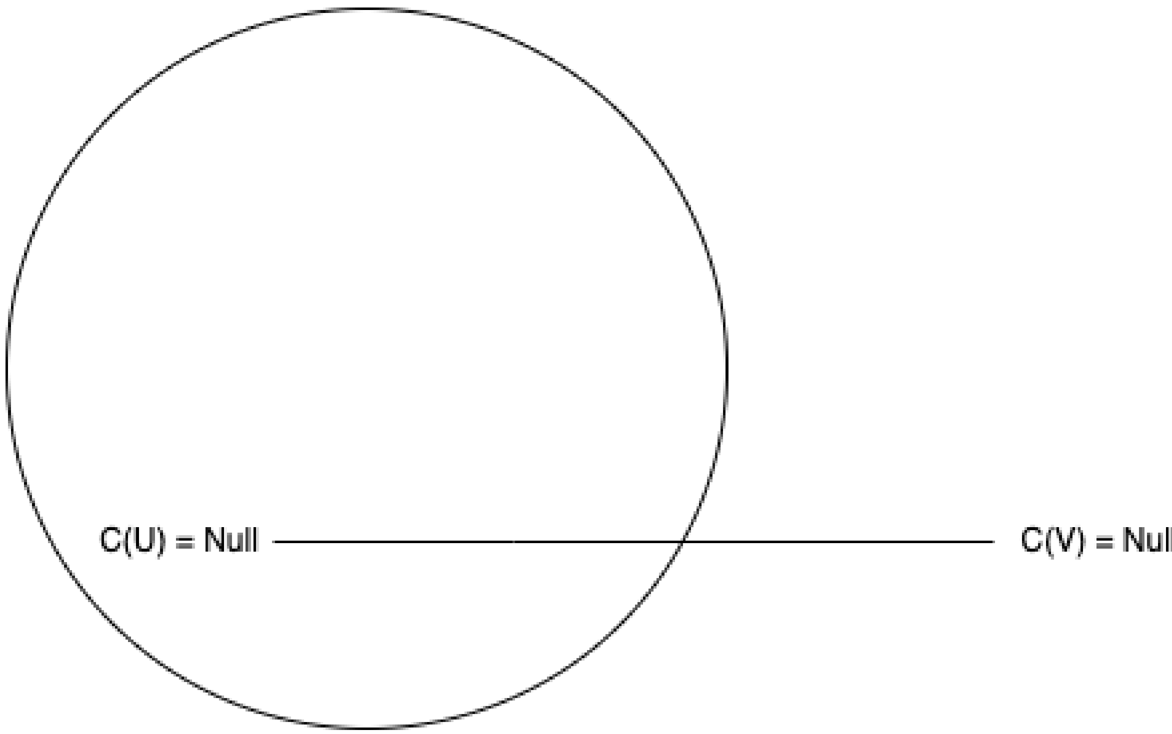

Figure 2:

P_r[Second condition holds] $\leq \frac{1}{j}$
If x_j is ranked fist among $x_1, x_2....x_j$, then probably x_j cuts edge e.

P_r[e is cut by $x - j$] = P_r[R $\in [L_{xj}, U_{xj}]$] and x_j is ranked first among $x_1, x_2....x_j$.
= P_r[R $\in [L_{xj}, U_{xj}]$] $P_r[x_j$ is first]

$\leq \frac{d(u,v)}{\Delta/4} \frac{1}{j}$

$= \frac{4d(u,v)}{\Delta j}$
If P_r[e is cut by some vertex]
$= \sum_{x_j} P_r[eiscutbyx_j]$

$\leq \sum_{j=1}^{n} \frac{4d(u,v)}{\Delta j}$

$\leq \frac{4d(u,v)}{\Delta j} O \log(n)$ (since $\sum_{j=1}^{n} \frac{1}{j} = \log(n)$)

Therefore Edge becomes crossing edge should have probability $\leq \frac{4d(u,v) O \log(n)}{\Delta j}$

3 Upcoming lecture:

- Claim: The distance $d_T(u,v) \geq d_g(u,v)$
- Proof: Base - If $d_T(u,v) \geq d_a(u,v)$ is a single vertex then $d_T(u,v) \geq d_a(u,v)$
- Induction hypothesis: $d_T(u,v) \geq d_a(u,v)$
- Induction: $[L_{xj}, U_{xj}]$ is given by $P_r[R \in [L_{xj}, U_{xj}]] \neq 4Da$. If x is ranked in π. Methods used to tackle this problem are going to be analyzed in the upcoming classes