CS 594: Representations in Algorithm Design **Spring 2022**

Lecture on 02/03/2022

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1 Last Lecture's Review

In the last lecture, we discussed algorithm to construct Low Stretch Tree for given weighted graph and the output will be distributions of all the trees (not a single tree).

- **Property:** All the distribution of tree should satisfy the following property,
  $\mathcal{T}$ - :For each tree T $\in \mathcal{T}$, $d_T(u,v) \geq d_a(u,v)$
  - $E_{T\in\mathcal{T}} \leq O(\log n \log \Delta) d_a(u,v)$
  This means that distance between pair of vertices might increase but expectation factor will not be more than $O(\log n \log \Delta) d_a(u,v)$

2 Bartal's Algorithm [Bartal in 1996]

Last lecture we discussed the proof of correctness of the algorithm based on the New LDD for the graph
Algorithm of New LDD : Assume given graph G with diameter Δ
Step 1: to choose parameter R - randomly sampled from [$\Delta/4, \Delta/2$]
Step 2: Random Permutation on all the vertices denoted by π
Goal : Assign each vertex with a color.
Initially no vertex is colored. C(V) $\leftarrow$ Null
for each vertex i =1 to n
if vertices such that C(v) = NULL and $d_a(\pi(i), v) \leq R$ then,
C(V) = i
within each class dist(u,v) $\leq \Delta$

Lemma: For each edge e= (u,v) P_r [e is an crossing edge] $\leq \frac{(d_a(v,u) \log n}{\Delta}$
Note: Distance on edge is large, then probability that edge e will bad is very high and vice versa
Observation: When e becomes an crossing edge ,

$\rightarrow C(u) \neq$ C(v)
Assume C(u)= i, C(v) = j

1. **Observation** If edge e is an crossing edge, $dist_a(\Delta(i), u) \leq R, dist_a(\Delta(i), v) > R$

Claim If edge e is cut by vertex x, then R $\in [dist_a(x, u), dist_a(x, 0)]$
Assumption: fix e=(u,v)
For another vertex x,
$L_x = min(dist_a(x, u), dist_a(u, v))$
$U_x = L_x + dist_a(u, v)$
The claim is equivalent to $R \in [L_x, U)$ when an edge is cut by X, then one necessary condition

1. If R < x, then u and v are both outside of ball of x, so x cannot assign values to u and v.

2. If R > x, then u and v are contained inside the ball of x so x can assign the colors and u ,v will be assigned same color by x.

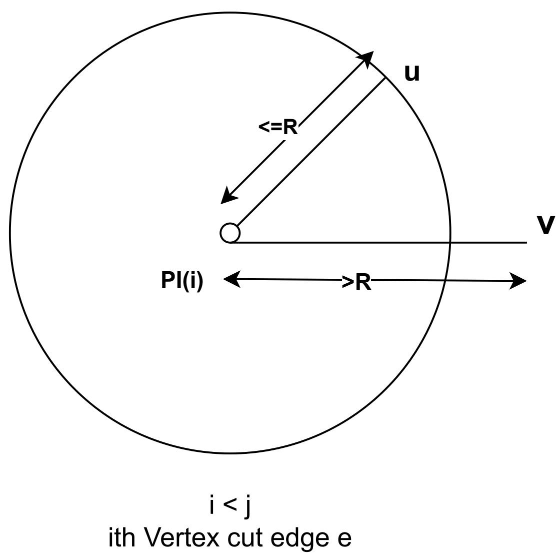

Figure 1:

Consider all the vertices that can potentially cut the edges here we take e = (u,v)
We consider L_x, U_x for the vertices and sort them according to L_x, U_x
1. For vertex X_j, L_{xj} and U_{xj}, if it overlaps in the interval($\Delta/4, \Delta/2$, then vertex x_2

or x_n cannot cut edge e.
2. For X_j, L_{xj} and U_{xj}, if the $\Delta/4 > L_xj, U_xj < \Delta/2$,
what is necessary condition that x_j cut edge e for permutation π
Suppose we x_k and x_j where $k < j$
If Rank of x_k < Rank of x_j , then x_j cannot cut edge e as x_k have smaller $L_{xk} < L_{xj}$ that is x_k has potential of cutting edge e. and one of two vertices u,v , one of u,v $\in B(x_k, R)$
Once the x_k is processed,one of u,v will be colored most likely, both of the vertices will be colored by then, so x_j will not have a chance to color the vertices.

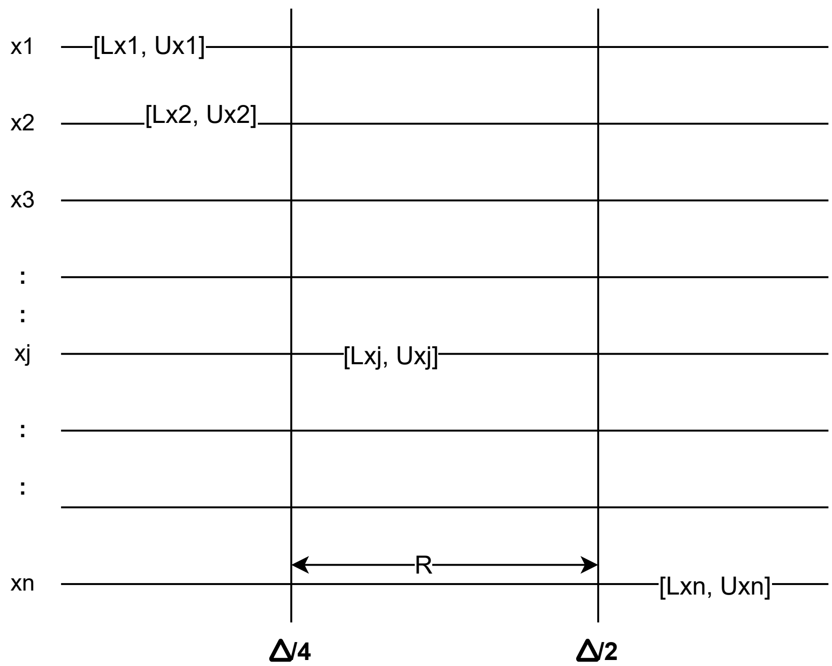

Figure 2:

Necessary Condition: For X to cut edge e in permutation π, then $\pi(x_k) > \pi(x_j)$ for all k < j.
Observation: For x_j to cut edge e, following two conditions should be satisfied,

1. R $\in [L_xj, U_xj]$
2. k < j , $\pi^{-1}(x_k) > \pi^{-1}(x_j)$ i.e. Rank of x_k > Rank of x_j

Based on 2 observations , find the upper bound of P_r[e is cut by x_j]
For a Random sample R in the range $(\Delta/4, \Delta/2)$, the probability that R falls in between $[L_{xj}, U_{xj}]$ is given by $P_r[R \in [L_{xj}, U_{xj}]] \leq \frac{d(u,v)}{\Delta/4}$
P_r[Second condition holds] $\leq \frac{1}{j}$
If x_j is ranked fist among $x_1, x_2....x_j$, then probably x_j cuts edge e.

P_r[e is cut by $x - j$] = P_r[R $\in [L_{xj}, U_{xj}]$] and x_j is ranked first among $x_1, x_2....x_j$.
= P_r[R $\in [L_{xj}, U_{xj}]$] $P_r[x_j$ is first]

$\leq \frac{d(u,v)}{\Delta/4}\frac{1}{j}$

$= \frac{4d(u,v)}{\Delta j}$
If P_r[e is cut by some vertex]
$= \sum_{x_j} P_r[eiscutbyx_j]$

$\leq \sum_{j=1}^{n} \frac{4d(u,v)}{\Delta j}$

$\leq \frac{4d(u,v)}{\Delta j} O\log(n)$ (since $\sum_{j=1}^{n} \frac{1}{j} = \log(n)$)

Therefore Edge becomes crossing edge should have probability $\leq \frac{4d(u,v)O\log(n)}{\Delta j}$

Thus discussion about LDD comes to an end.

3 Graph Representations

Idea: Concept of well-connected and how to preserve the property of the graph that is well connected.

Well-Connected
Graph G is well connected,
In graph g, random walk from vertex S to vertex s is O(n) expectation.

Random Walk on Graph G
Consider a graph G and person P stands at vertex x initially. The person can move random vertex in random way. This walk doesn't have specified destination.So At each step P randomly chooses the next step. In next step, Person P at vertex 1 has

probability $\frac{1}{4}$ because P has 4 neighbors. The choice of current step does not depend on choice of previous step. And also ha no implication in future steps
So Random decision at each step is totally independent. **Random Walk on weighted**

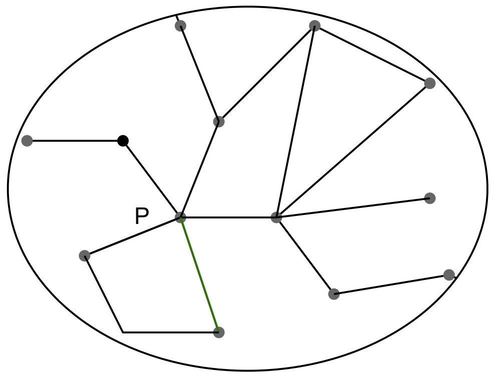

Figure 3: Graph G

graph: In weighted graph, the random walk choose neighbors according to edge weights.

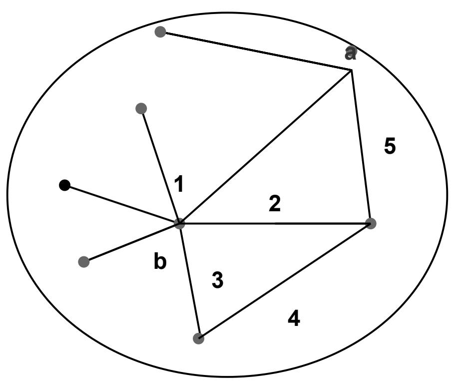

Figure 4: Random Walk on weighted graph G

Random walk from vertex a to next step
with probability $\frac{1}{1+5} = \frac{1}{6}$
Moves to b
with probability $\frac{5}{1+5} = \frac{5}{6}$
Moves to c

with probability $\frac{5}{5+2+6} = \frac{1}{13}$
Moves to a
with probability $\frac{2}{13}$
Moves to b
with probability $\frac{6}{13}$
Moves to d
Example: Graph g is complete graph.
Claim: Complete graph is a well connected graph
Proof: After one step walk, P_r[At vertex v] $= \frac{1}{n}$
For all vertices in graph, after one step walk, the probability that walk to arbitrary two vertices have the same chance.
This condition continues for 2 step walk, 3 step walk and so on..

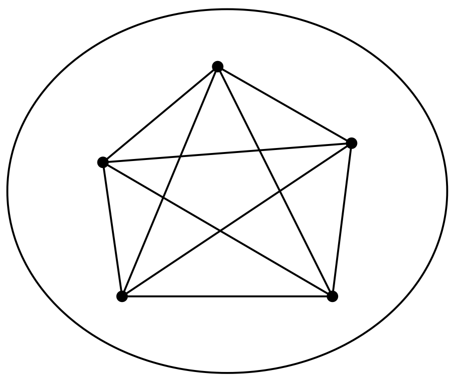

Figure 5: A Complete Graph

We will discuss about the definition of well connected graph in next lecture.