CS 594: Representations in Algorithm Design	**Spring 2022**

Lecture on 02/08/2022

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1 Last Lecture's Review

In the last lecture, we discussed Bartal's algorithm. While we analysed Bartal's Algorithm we were able to come up with the following conclusion as shown below.

We also had some early discussions about Graph Representations which entailed Well-Connected and Random Walks on Graphs.

- **Conclusion:**
  The Edge which becomes crossing edge should have probability $\leq \frac{4d(u,v)O\log(n)}{\Delta j}$

2 Well-Connected Graph

The graph expansion which takes place is constant and the expansion rate can be found out to be propotional to $\frac{1}{n^2}$

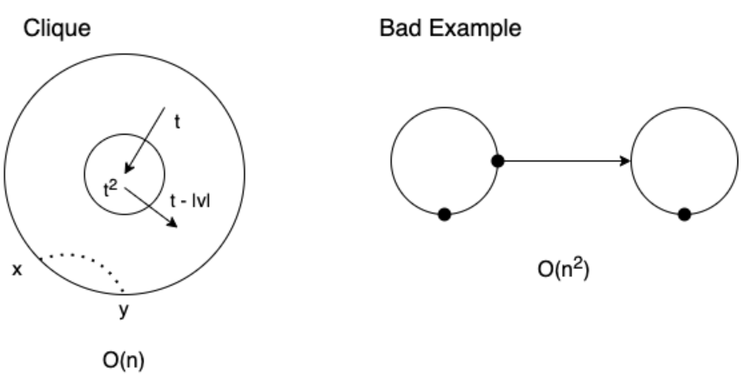

Figure 1: Graph Well Connectivity

Expander: In an expander most edges are moving outside rather than staying inside.

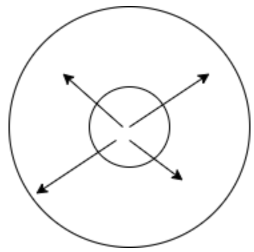

Figure 2: Expander

Edge Expansion:

$$\min_{s \leq v} \frac{E(S, \bar{S})}{\min\{Vol(S), Vol(\bar{S})\}^2} \leq 1 \tag{1}$$

$$volume = Vol(S) = \sum_{v \in s} d(v) \tag{2}$$

If the graph has a large expansion then it is a good expander. Note that expansion of a graph is constant

Random Walk: Looks at the surrounding neighbours and choose an arbitrary neighbour (which is propotional to the edge weight)

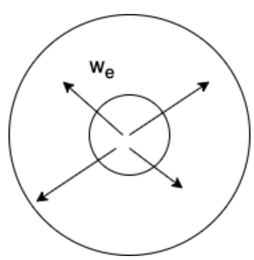

Figure 3: Expander

Hitting Time of Random Walk: It can be defined as the estimated time of the first visit to vertex ”y” from the vertex ”x”.

$$G = (V, E, W),\ vertex(u) \tag{3}$$

This is to define the hitting time to the vertex(u).

x_v to denote E time to the first visit to to the vertex v

$$v = u \rightarrow x_v = 0 \tag{4}$$

$$v \neq u \rightarrow x_v = 1 + \sum_{v_i \sim v} \frac{w_{vv_i}}{d_v}.x_{v_i} \tag{5}$$

Used to compute 1st visiting for the vertex

$$d_v.x_v = d_v + \sum_{v_i \sim v} w_{vv_i}.x_{v_i} \tag{6}$$

3 Graph Laplasian Matrix

Now we will discuss about the graph laplasian matrix.

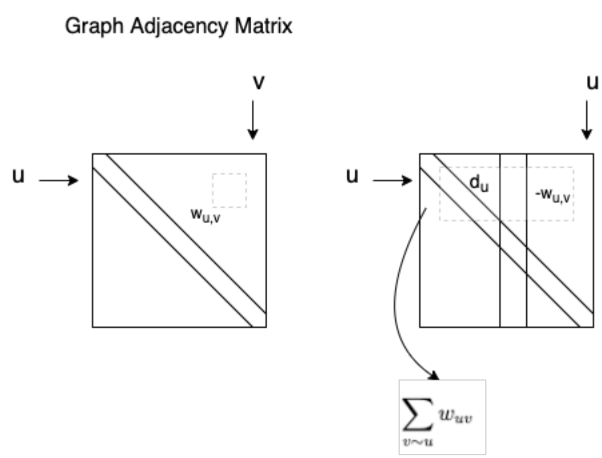

Figure 4: Graph Adjacency Matrix

$$\sum_{v \in V} L_{u.v} = 0 \text{ for any u} \tag{7}$$

In the following L(G) is the laplasian function for a graph "G".

Symmetric:

$$L_{uv}(G) = L_{vu}(G) \tag{8}$$

Positive semidefinite: λ of L(G) ≥ 0

Eigen: 0 is the eigen value of L(G)

$$L(G).\vec{\mathbb{1}} = \vec{0} \tag{9}$$

$$L(G) = \sum_{i=1}^{n} \lambda_i.u_i.{u_i}^T \tag{10}$$

$$\lambda_i \geq 0 \text{ and } \lambda_i = 0, u_1 = \mathbb{1} \tag{11}$$

$$M_{u_i} = \lambda_i.u_i \tag{12}$$

$$L(G).\mathbb{1} = \vec{0} \text{ and } L(G).x \neq y \tag{13}$$

If $\lambda_i \neq 0$ then we could say that,

$$u_i \perp u_1 \Rightarrow L(G).u_i = \lambda_i.u_i \neq 0 \tag{14}$$

Now,

$$v \neq u \tag{15}$$

$$L_u(G).X = d_u \tag{16}$$

$$L_u(G).X = d - 2m.1u \tag{17}$$

Where 2m.1u is a vector and can be denoted by "z" and that z $\perp \vec{\mathbb{1}} = 0$

Now we can find the pseudo-inverse,

$$x_u = 0 \tag{18}$$

$$y = [L(G)]^{-1}.(d - 2m.1u) \text{ where (d-2m.1u } \perp \mathbb{1}) \tag{19}$$

Relation between hitting time and graph expansion:

If,

$$\vec{y} = [L(G)]^{-1}.(d - z) \tag{20}$$

Then the hitting time for v can be said to be,

$$(1v - 1u)^T.y \Rightarrow (1v - 1u)^T.L^T.(d - 2m.\mathbb{1}u) \tag{21}$$

Now the hitting time from v to u and vice versa can be found out to be,

$$(1v - 1u)^T.L^T.(d - 2m.1u) + (1u - 1v)^T.L_T.(2m.1v - d) \tag{22}$$

$$(1v - 1u)^T.L^T.(2m.1v - 2m.1u) \tag{23}$$

$$2m.(1v - 1u)^T.L^T.(1v - 1u) \leq 4m\lambda_2^{-1} \leftarrow \sum \frac{w_{uv}}{2} \tag{24}$$

Equation (24) is related to the 2nd eigen value of the Laplasian matrix. If λ_2 is larger then the hitting time can be concluded to be smaller.

Theorem:

$$\frac{1}{2}.\lambda_2(D_a^{-1}L(G).D_a^{-1/2}) \leq \Phi(G) \leq \sqrt[4]{\lambda_2}(D_a^{-1/2}L(G).D_a^{1/2}) \tag{25}$$

Equation (25) depicts and defines Cheeger's inequality which we will talk about in the next class.