Dynamic connectivity

- Input changes (maintain a ds for some problem)
- Graph updates: edge insertions/deletions
  query (x, y):

Goal: Handle updates and answer query as fast as possible.

Thm: Dynamic connectivity in $O(\text{polylog } n)$ time for each update and query.

$(\log n)^2$

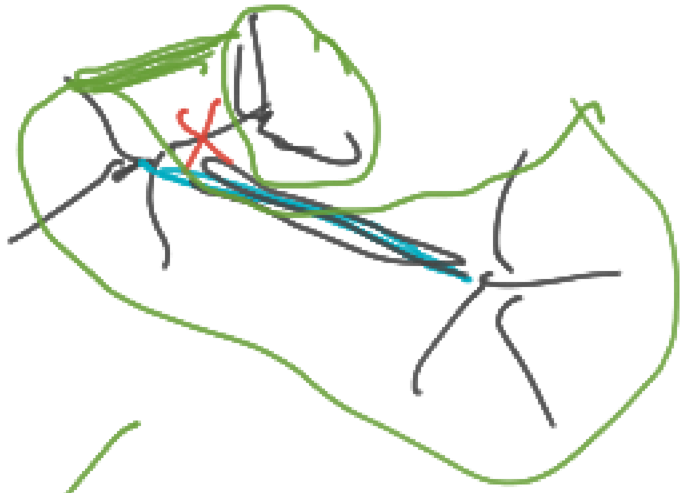
[Henzinger-King 1999]

Approach: Maintain a spanning ~~tree~~ forest of the graph

Observation: Each update requires at most one edge change in spanning forest.

- Insertion: ① edge contains two vertices in same CC ⇒ No update

  ② - - - - - - - - - - - - - different CCs ⇒ put new edge in spanning forest

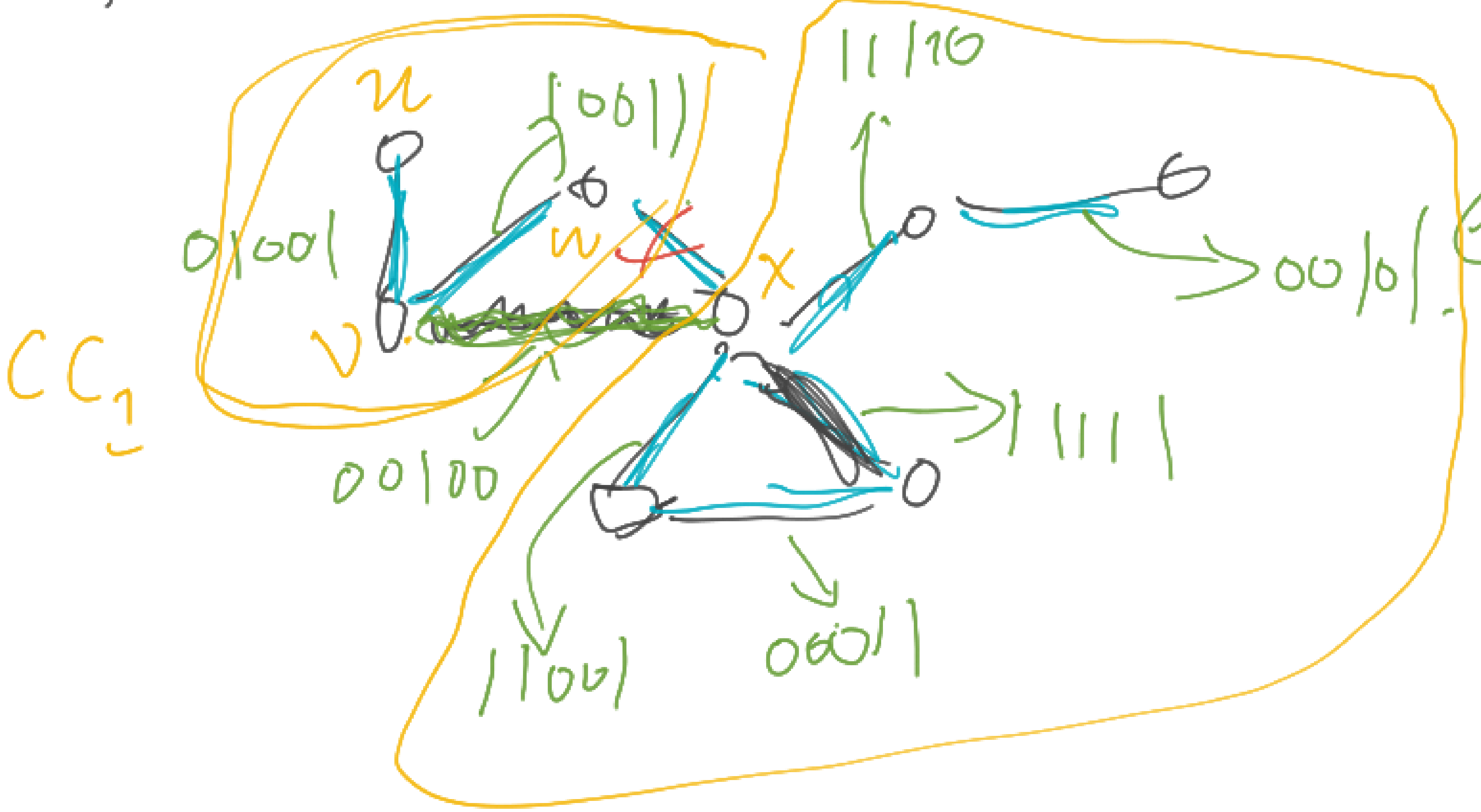


- Deletion: ③ deleted edge is not in spanning forest ⇒ No update.

  ④ - - - - - - - - - - - - - is in spanning forest. ⇒ Find replacement (If possible).

If replacement is unique, then easy

binary name.



$S_{CC_1} = S_u \oplus S_v \oplus S_w$

$= 01001 \oplus 11110 \oplus 10011$

$= 00100$

name of (v, x)

$S_u = 01001$

$S_v = 01001 \oplus 00100 \oplus 10011 = 11110$

$S_w = 10011$

If edge $(u, v) \xrightarrow{(01001)}$ connecting two vertices

both in CC_1

$S_{CC_1} = S_u \oplus S_v \oplus S_w$

$b \oplus b = 0$

If edge have two endpoints in difference

CC, (e.g. (v, x).)

$S_{CC_2} = S_u \oplus S_v \oplus S_w$

Conclusion:

- If a connected componet dose not have any outgoing edge, $S_{cc} = 0$

edge with one endpoint in cc
one . . . . outside

- If only one edge is outgoing
S_{cc} = name of the outgoing edge.

?- If more than one outgoing edge.
S_{cc} = $\oplus$ names of all outgoing edges.

IDEA: Make replacement unique

- Sample some edges for a new graph.

Assume graph have two CCs. there are t edges between them.

Sampled graph: foreach edge in G, the edge in Sampled graph (G_p) with probability $\frac{1}{t}$.

$\Pr[\text{only one outgoing edge is sampled in } G_p]$

$= t \cdot \frac{1}{t} \cdot (1 - \frac{1}{t})^{t-1} \rightarrow \frac{1}{e} \leftarrow$ a constant.

Q: How many outgoing edges?

A: We do not know

IDEA: Maintain sampled graphs with prob

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \frac{1}{n}$$ ← log n probabilities.

Claim: for a CC with t outgoing edges

$\exists$ sampled graph with p

$$\frac{1}{t} \geq p \geq \frac{1}{2t}.$$

$\Pr[$only one outgoing edge sampled$] \geq \frac{1}{e^2}$ ← constant.

Q: constant probability is not good

A: maintain many sampled graph

for each $p = \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{n}$.

claim: If ~~poly~~ $\log n$ sampled graphs are maintained

$\Rightarrow c \log n$ s

then with probability $1 - \frac{1}{n^{100}}$,

at least one sampled graph gives a replacement.

$\boxed{Scc}$

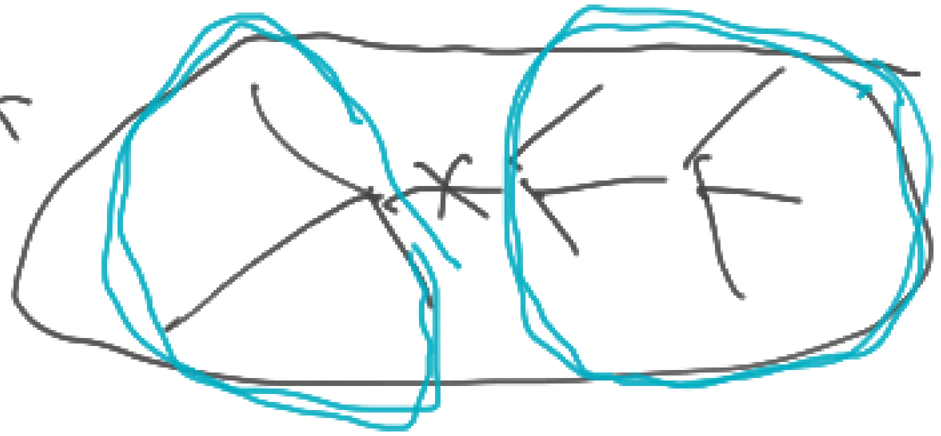
ET Tree [Henzinger – King 1999].

① link two trees together by adding a new edge.

② cut a tree ~~into~~ into two trees by deleting

③ maintain labels of vertices an edge, and label for a tree is sum of labels of vertices in the tree (label of vertex can change).

④. get root of the tree containing a given vertex



Data Structure:

- Spanning forest.
- Sampled graphs G_p ⟸ polylog many.
  ↳ ET tree with vertex label to be sum of name of incidence edges in G_p

Insert (x, y).

① Insert (x, y) to spanning tree if needed.

②. Maintain G_p: For each G_p, sample edge (x, y). with prob p, add to G_p

- Delete (x,y)

① Maintain G_p, if (x,y) is G_p.

then we delete it, (maintain ET tree)

② Find replacement: For each G_p,

Let CC_2 be the CC of x.

use ET for G_p to get S_{CC_1},

check if S_{CC_2} is an edge name

if yes, edge is a replacement.

- Query (x,y): root of x, y (r_x, r_y)

Yes if $r_x = r_y$, No otherwise.