Differential Geometry Definitions and Such

based on Thorpe, J.A.,*Elementary Topics in Differential Geometry*, Springer-Verlag, New York, NY, 1979.

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1 Graphs and Level Sets

Image A set of values given by a mathematical function corresponding to a particular subset of the domain.

2 Vector Fields

Vector a *pair* $\mathbf{v} = (p, v)$ where p defines a point $p \in \Re^{n+1}$, and v defines a direction and magnitude in $\Re^{n+1}$.

Vector Field A vector field $\mathbf{X}$ on $U \subset \Re^{n+1}$ is a function which assigns to each point of U a vector at that point. $\mathbf{X}(p) = (p, X(p))$

Smooth A function $f : U \rightarrow \Re$ (U an open set in $\Re^{n+1}$) is smooth if all its partial derivatives of *all* orders exist and are continuous. (A vector function is smooth if all its component functions are smooth.)

Parameterized Curve in $\Re^{n+1}$ is a smooth function $\alpha : I \rightarrow \Re^{n+1}$, where I is some open interval in $\Re$. Basically, it assigns points in the space as a function of time.

$$\alpha(t) = (x_1(t), x_2(t), \cdots, x_{n+1}(t))$$

Velocity Vector at time $t \in I$ of the parameterized curve $\alpha : I \rightarrow \Re^{n+1}$ is the vector at $\alpha(t)$ defined by

$$\dot{\alpha}(t) = \left(\alpha(t), \frac{d\alpha}{dt}(t)\right) = \left(\alpha(t), \frac{dx_1}{dt}(t), \cdots, \frac{dx_{n+1}}{dt}(t)\right)$$

This vector is tangent to the curve $\alpha(t)$

Integral Curve A parameterized curve $\alpha : I \rightarrow \Re^{n+1}$ is said to be an *integral curve* of the vector field $\mathbf{X}$ on the open set U in $\Re^{n+1}$ if $\alpha(t) \in U$ and $\dot{\alpha}(t) = \mathbf{X}(\alpha(t)) \ \ \forall t \in I$.

Divergence of a smooth vector field $\mathbf{X}$ on $U \in \Re^{n+1}$

$$\mathbf{X}(p) = (p, X_1(p), \cdots, X_{n+1}(p)) \text{ for } p \in U$$

is the function $div\mathbf{X} : U \rightarrow \Re$ defined by

$$div\mathbf{X} = \Sigma_{i=1}^{n+1}(\partial X_i/\partial x_i)$$

Complete A smooth vector field $\mathbf{X}$ on an open set U of $\Re^{n+1}$is said to be *complete* if for each $p \in U$ the maximal integral curve of $\mathbf{X}$ has domain equal to $\Re$.

3 The Tangent Space

Tangent A vector at p is said to be *tangent to the level set* $f^{-1}(c)$:if it is a velocity vector of a parameterized curve in $\Re^{n+1}$ whose image is contained in $f^{-1}(c)$:.

Regular Point A point $p \in \Re^{n+1}$ such that $\nabla f(p) \neq 0 \ \forall \ p \in \Re^{n+1}$.

4 Surfaces

Surface of dimension n, or n-surface, in $\Re^{n+1}$is a non-empty subset S of $\Re^{n+1}$of the from $S = f^{-1}(c)$ where $f : U \to \Re$, U open in $\Re^{n+1}$, is a smooth function with the property that $\nabla f(p) \neq 0 \ \forall \ p \in S$.

$$S = \{\mathbf{x} : f(\mathbf{x}) = c\}$$

Tangent Space an n-dimensional subpace of the vector space $\Re_p^{n+1}$ consisting of all vectors tangent to the surface S at p. The tangent space, denoted S_p, is dependent only on S and not on the function used to define S.

Compact A surface $S \in U$ is compact (closed and bounded) if every smooth function $g : U \to \Re$ attains a maximum and a minimum on S.

5 Vector Fields on Surfaces

Connected A subset $S \subset \Re^{n+1}$ is said to be connected if for each pair p, q of points in S there is a continuous map $\alpha : [a, b] \to S$, from some closed interval $[a, b]$ into S such that $\alpha(a) = p$ and $\alpha(b) = q$. Thus S is connected if each pair of points in S can be joined by a continuous, but not necessarily smooth, curve which lies completely in S.

Orientation A smooth unit normal vector field on S in $\Re^{n+1}$. (An n-surface, together with an orientation, is called an oriented n-surface.)

Direction A unit vector in $\Re^{n+1}$at $p \in \Re^{n+1}$ is called a direction at p.

Consistent An ordered basis , given as $\{v_1, v_2, \ldots, v_n\}$ (not necessarily orthonormal), for the tangent space S_p at p of the oriented n-surface S is *consistent* with the orientation $\mathbf{N}$ on S if the determinant

$$\det \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \\ \mathbf{N} \end{pmatrix} > 0$$

6 The Gauss Map

Gauss Map A smooth map $N : S \to \mathbf{S}^n$ associated with each oriented n-surface. Translates all unit normal vectors on a surface to the origin. The image of the Gauss map is called the spherical image of the oriented n-surface S.

7 Geodesics

Geodesic A parameterized curve $\alpha : I \to S$ in an n-surface $S \subset \Re^{n+1}$ whose acceleration is everywhere orthogonal to S; that is, $\ddot{\alpha}(t) \in S_p^{\perp} \;\; \forall\, t \in I$. Thus a geodesic is a curve in S which always goes "straight ahead" in the surface, with no component of acceleration tangent to the surface.

Vector Field along a parameterized curve A vector field $\mathbf{X}$ along a parameterized curve $\alpha : I \to \Re^{n+1}$ is a function which assigns to each $t \in I$ a vector $\mathbf{X}(t)$ at $\alpha(t)$; i.e., $\mathbf{X}(t) \in \Re^{n+1}_{\alpha(t)} \;\; \forall\, t \in I$.

Function along a parameterized curve A function f along a parameterized curve $\alpha : I \to \Re^{n+1}$ is simply a function $f : I \to \Re$.

Speed $\|\dot{\alpha}(t)\|$ is called the *speed* of $\alpha(t)$.

8 Parallel Transport

Tangent Vector Field A vector field $\mathbf{X}$ along a parameterized curve $\alpha : I \to S$ in an n-surface S is tangent to S along α if $\mathbf{X}(t) \in S_{\alpha(t)} \;\; \forall t \in I$.

Covariant Derivative of $\mathbf{X}$ is the vector field $\mathbf{X}'$ tangent to S along α defined by

$$\mathbf{X}^{'}(t) \;=\; \dot{\mathbf{X}}(t) - \left[\dot{\mathbf{X}}(t) \cdot \mathbf{N}\left(\alpha\left(t\right)\right)\right] \mathbf{N}(\alpha(t))$$

Euclidean Parallel Two vectors $\mathbf{v} = (p, v)$ and $\mathbf{w} = (p, w)$ are Euclidean parallel if $v = w$.

Levi-Civita Parallel A smooth vector field $\mathbf{X}$ tangent to S along α is Levi-Civita parallel, or just parallel, if $\mathbf{X}^{'} = 0$.

Parallel Transport Each parameterized curve $\alpha : [a, b] \to S$ from point p to point q determines a map $P_\alpha : S_p \to S_p$ by

$$P_\alpha(\mathbf{v}) = \mathbf{V}(b)$$

where for $\mathbf{v} \in S_p$, $\mathbf{V}$ is a unique parallel vector field along α with $\mathbf{V}(a) = \mathbf{v}$. $P_\alpha(\mathbf{v})$ is called the *parallel transport* of $\mathbf{v}$ along α to q.

Piecewise Smooth A *piecewise smooth parameterized curve* α in S is a continuous map $\alpha : [a, b] \to S$ such that the restriction of α to $[t_i, t_{i+1}]$ is smooth for each $i \in \{0, 1, \ldots, k\}$, where $a = t_0 < t_1 < \cdots < t_{k+1} = b$.

9 The Weingarten Map

Derivative Given a smooth function f defined on an open set U in $\Re^{n+1}$and a vector $\mathbf{v} \in \Re_p^{n+1}$, $p \in U$, the derivative of f with respect to $\mathbf{v}$ is the real number

$$\nabla_{\mathbf{v}} f = (f \circ \alpha)'(t_o)$$

where $\alpha : I \to U$ is *any* parameterized curve in U with $\dot{\alpha}(t_o) = \mathbf{v}$. Note, by chain rule

$$\nabla_{\mathbf{v}} f = (f \circ \alpha)'(t_o) = \nabla f(\alpha(t_o)) \cdot \dot{\alpha}(t_o) = \nabla f(p) \cdot \mathbf{v}$$

The derivative is a linear map which sends $\mathbf{v}$ into $\nabla_{\mathbf{v}} f$ from $\Re^{n+1} \to \Re$.

Directional Derivative If $\|\mathbf{v}\| = 1$, then $\nabla_{\mathbf{v}} f$ is called the *directional derivative* of f at p in the direction $\mathbf{v}$.

Derivative of Vector Field The *derivative of a smooth vector field* $\mathbf{X}$ on an open set U in $\Re^{n+1}$with respect to a vector $\mathbf{v} \in \Re_p^{n+1}$,$p \in U$ is defined by

$$\nabla_{\mathbf{v}} \mathbf{X} = (\mathbf{X} \dot{\circ} \alpha)(t_0)$$

where $\alpha : I \to U$ is *any* parameterized curve in U such that $\dot{\alpha}(t_o) = \mathbf{v}$

If $\mathbf{X}$ is a smooth vector field on an n-surface S in $\Re^{n+1}$and $\mathbf{v}$ is a vector tangent to S at $p \in S$, the formula is the same but now α is required to be a parameterized curve in S with $\dot{\alpha}(t_0) = \mathbf{v}$

Either way,

$$\begin{aligned} \nabla_{\mathbf{v}} \mathbf{X} = &\quad (\alpha(t_0), (X_1 \circ \alpha)'(t_0), \ldots, (X_{n+1} \circ \alpha)'(t_0)) \\ = &\quad (p, \nabla_{\mathbf{v}} X_1, \ldots, \nabla_{\mathbf{v}} X_{n+1}) \end{aligned}$$

Note, $\nabla_{\mathbf{v}} \mathbf{X}$ is a linear map, from $\Re_p^{n+1}$ into $\Re_p^{n+1}$ if $\mathbf{X}$ is defined on U, or from S_p into $\Re_p^{n+1}$ if $\mathbf{X}$ is defined on an n-surface S.

Covariant Derivative of the *tangent* vector field $\mathbf{X}$ with respect to $\mathbf{v} \in S$, is defined by

$$D_{\mathbf{v}} \mathbf{X} = \nabla_{\mathbf{v}} \mathbf{X} - (\nabla_{\mathbf{v}} \mathbf{X} \cdot \mathbf{N}(p))\, \mathbf{N}(p)$$

where $\mathbf{N}$ is an orientation on S.

Note, $D_{\mathbf{v}} \mathbf{X} = (\mathbf{X} \circ \alpha)'(t_o)$ where $\alpha : I \to S$ is *any* parameterized curve in S with $\dot{\alpha}(t_o) = \mathbf{v}$.

For a smooth tangent vector field $\mathbf{X}$ on S, the function which sends $\mathbf{v}$ into $D_{\mathbf{v}} \mathbf{X}$ is a linear map from S_p into S_p.

Weingarten Map of S at p, is a linear map $L_p : S_p \to S_p$ defined by

$$\begin{aligned} L_p(\mathbf{v}) = &\quad -\nabla_{\mathbf{v}} \mathbf{N} \\ = &\quad -(\mathbf{N} \dot{\circ} \alpha)(t_o) \end{aligned}$$

where $\alpha : I \to S$ is *any* parameterized curve in S with $\dot{\alpha}(t_o) = \mathbf{v}$.

The Wiengarten Map, $L_p(\mathbf{v})$, measures (up to a sign), the rate of change of $\mathbf{N}$ as one passes through p along any curve α . Since the tangent space is orthogonal to $\mathbf{N}$, this is also a measure of the turning of the tangent space $\Longrightarrow$ *shape operator of* S.

10 Curvature of *Plane* Curves

For this section we are restricted to plan curves, $C = f^{-1}(c)$ where $f : U \to \Re$, in the open set $U \subset \Re^2$. These curves are oriented by $\mathbf{N} = \nabla f / \|\nabla f\|$.

Curvature A real number, κ, such that

$$L_p(\mathbf{v}) = \kappa(p)\mathbf{v}$$

Since $L_p(\mathbf{v})$ is a linear transformation on the 1-D space C_p, to the same space, it is in fact a scalar multiplication.

$$\kappa(p) = L_p(\mathbf{v}) \cdot \mathbf{v} / \|\mathbf{v}\|^2$$

If $\alpha : I \to C$ is any parameterized curve in C with $\dot{\alpha}(t) \neq 0 \ \ \forall\, t \in I$, then by theorem 1 of chapter 9

$$\kappa\left(\alpha\left(t\right)\right) = \frac{L_p\left(\dot{\alpha}\left(t\right)\right) \cdot \dot{\alpha}\left(t\right)}{\|\dot{\alpha}\left(t\right)\|^2} = \frac{\ddot{\alpha}\left(t\right) \cdot \mathbf{N}\left(\alpha\left(t\right)\right)}{\|\dot{\alpha}\left(t\right)\|^2}$$

Parameterization of a segment of C containing p is a parameterized curve $\alpha : I \to C$ which

i) is regular; i.e., has $\dot{\alpha}(t) \neq 0 \ \ \forall\, t \in I$

ii) is oriented consistently with C; i.e., is such that for each $t \in I$ the basis $\{\dot{\alpha}(t)\}$ for $C_{\alpha(t)}$ is consistent with the orientation $\mathbf{N}$ of C, and

iii) has $p \in \text{Image}\,\alpha$.

Global Parameterization If α is onto; i.e., if $\alpha\left(I\right) = C$, α is called a *global parameterization* of C. Otherwise, α is a *local parameterization.*

Circle of Curvature For C an arbitrary oriented plane curve and $p \in C$ such that $\kappa\left(p\right) \neq 0$, there exists a *unique* oriented circle $\mathbf{O}$, called the *circle of curvature* of C at p, which

i) is tangent to C at p (i.e., $C_p = \mathbf{O}_p$).

ii) is oriented consistently with C (i.e.,$\mathbf{N}(p) = \mathbf{N}_1(p)$, where $\mathbf{N}$ and $\mathbf{N}_1$ are respectively the orientation normals of C and $\mathbf{O}$), and

iii) whose normal turns at the same rate at p as does the normal to C (i.e.,$\nabla_{\mathbf{v}}\mathbf{N} = \nabla_{\mathbf{v}}\mathbf{N}_1 \ \ \forall\, \mathbf{v} \in C_p = \mathbf{O}_p$).

Radius of Curvature The radius of the *circle of curvature*, which is given by $r = 1/\|\kappa(p)\|$ as determined by condition (iii) above.

Center of Curvature The center of the *circle of curvature* at p.

11 Arc Length and Line Integrals

Length , $l(\alpha)$ of a parameterized curve $\alpha : I \to \Re^{n+1}$ is defined to be the integral of the speed of α:

$$l(\alpha) = \int_a^b \|\dot{\alpha}(t)\| dt$$

where a and b are the endpoints of I (possibly at $\pm\infty$). If α is a unit speed parameterization, then $l(\alpha) = b - a$, and the curve is said to be parameterized by *arc length*.

Period of a periodic function β is the smallest τ such that $\beta(t + \tau) = \beta(t)$ for all such t such that both t and $t + \tau$ are in the domain of β.

Fundamental Domain of periodic function β is the half-open interval $[t_0, t_0 + \tau)$.

Differential 1-form , or simply a 1-form, on an open set $U \subset \Re^{n+1}$ is a function $\omega : U \times \Re^{n+1} \to \Re$ such that, for each $p \in U$, the restriction of ω to $\Re^{n+1}_p \subset U \times \Re^{n+1}$ is linear. (It maps a vector $\mathbf{v} = (p, v)$ to a scalar.)

Dual to X is the 1-form ω_x on U, defined as

$$\omega_x(p, v) = \mathbf{X} \cdot (p, v)$$

where **X**is a vector field on U.

Differential of f on U, denoted df, where $f : U \to \Re$ is a smooth function, is defined as

$$df(\mathbf{v}) = \nabla_{\mathbf{v}} f = \nabla f(p) \cdot \mathbf{v} \quad (\mathbf{v} = (p, v) \in \Re^{n+1}_p,\ p \in U)$$

Integral of ω over α is the real number

$$\int_\alpha \omega = \int_a^b \omega\left(\dot{\alpha}\ \ (t)\right) dt$$

Integrals of this type are called *line integrals*. For a reparmeterization of α given by β,

$$\int_\beta \omega = \int_\alpha \omega$$

Exact A 1-form which is the differential of a smooth function is said to be *exact*.

Closed A parameterized curve $\alpha : [a, b] \to \Re^{n+1}$ with $\alpha(a) = \alpha(b)$ is said to be *closed*. The integral of an exact 1-form over a closed curve is always zero. (Think of mechanical work as a conservative function.) The integral of an exact 1-form over a compact connection oriented plane curve is always zero.

Winding Number The integer given by

$$k(\alpha\ ) = \frac{1}{2\pi} \int_\alpha \eta$$

, where
$$\eta = -\frac{x_2}{x_1^2 + x_2^2} dx_1 + \frac{x_1}{x_1^2 + x_2^2} dx_2$$
for $\alpha : [a, b] \to \Re^2 - \{0, 0\}$.

The winding number counts the number of times that a curve α winds around the origin.

12 Curvature of Surfaces

Normal Curvature of S at p in the direction $\mathbf{v}$ is given by

$$k(\mathbf{v}) = L_p(\mathbf{v}) \cdot \mathbf{v}$$

This is the normal component of acceleration for all parameterized curves α in S passing through p with velocity $\mathbf{v}$. If $k(\mathbf{v}) > 0$ the surface bends toward the normal $\mathbf{N}$. If $k(\mathbf{v}) < 0$ then the surfaces bends away from $\mathbf{N}$ in the direction $\mathbf{v}$.

For $n = 1$, $k(\mathbf{v}) = \kappa(p)$ for both unit vectors $\mathbf{v} \in S_p$.

Normal Section a plane section cut from the surface. The plane is determined by the vectors $\mathbf{v}$ and $\mathbf{N}$.

Principal Curvatures are the eigenvalues of the Weingarten map.

Principal Curvature Directions are the eigenvectors of the Weingarten map.

Direction Cosines of v are given by

$$\cos(\Theta_i) = \mathbf{v} \cdot \mathbf{v}_i$$

where $\mathbf{v}_i$ is the i^{th} principal curvature direction.

The normal curvature $k(\mathbf{v})$ in the direction $\mathbf{v} \in S_p(\|\mathbf{v}\| = 1)$ is given by

$$k(\mathbf{v}) = \sum_{i=1}^{n} k_i(p) \left(\mathbf{v} \cdot \mathbf{v}_i\right)^2 = \sum_{i=1}^{n} k_i(p) \cos^2 \Theta_i$$

Quadratic Form of L (where L is a self adjoint linear transform), denoted $\mathcal{Q}$, is given by

$$\mathcal{Q}(v) = L(v) \cdot v$$

where $L : V \to V$, with V being a vector space with a dot product. The quadratic form, $\mathcal{Q}$, is a real valued function $\mathcal{Q} : V \to \Re$.

Note, $\mathcal{Q}$contains the same information as L since

$$L(v) \cdot w = \frac{1}{2} \left[\mathcal{Q}(v + w) - \mathcal{Q}(v) - \mathcal{Q}(w)\right]$$

, which is valid for all v and w in V.

Definite A quadratic form $\mathcal{Q}$is said to be

positive definite if $\mathcal{Q}(\mathbf{v}) > 0 \ \ \forall \mathbf{v} \neq \mathbf{0}$

negative definite if $\mathcal{Q}(\mathbf{v}) < 0 \ \ \forall \mathbf{v} \neq \mathbf{0}$

definite if $\mathcal{Q}$is either positive or negative definite $(\forall \mathbf{v} \mathcal{Q}(\mathbf{v}) \neq 0)$

indefinite if $\mathcal{Q}$is neither positive or negative definite $(\exists \mathbf{v} : \mathcal{Q}(\mathbf{v}) = 0)$

positive semi-definite if $\mathcal{Q}(\mathbf{v}) \geq 0 \ \ \forall \mathbf{v} \neq \mathbf{0}$

negative definite if $\mathcal{Q}(\mathbf{v}) \leq 0 \ \ \forall \mathbf{v} \neq \mathbf{0}$

semi-definite if $\mathcal{Q}$is either positive or negative semi-definite

First Fundamental Form of S at p is the quadratic form $\mathcal{F}_p$associated with the identity transformation on S_p, given by

$$\mathcal{F}_p(\mathbf{v}) = \mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 \ , \ \ \forall \mathbf{v} \in S_p$$

The $\mathcal{F}_p$is always positive definite.

Second Fundamental Form of S at p, denoted $\mathcal{L}_p$, is the quadratic form associated with the Weingarten map given by

$$\mathcal{L}_p(\mathbf{v}) = L(\mathbf{v}) \cdot \mathbf{v} = \ddot{\alpha} \ (t_0) \cdot \mathbf{N}(p)$$

where $\alpha \ : I \to S$ is any parameterized curve in S with $\alpha \ (t_0) = p$ and $\dot{\alpha} \ (t_0) = \mathbf{v}$.

When $\|\mathbf{v}\| = 1$, $\mathcal{L}_p(\mathbf{v})$ is equal to the normal curvature of S at p in the direction $\mathbf{v}$.

Gauss-Kronecker Curvature of S at p, denoted $K(p)$ is equal to the product of the principal curvatures at p, or

$$K(p) = \det L_p$$

Gaussian Curvature when n=2, $K(p) = k_1(p)k_2(p)$.

Mean Curvature of S at p, denoted $H(p)$, is given by

$$H(p) = \frac{1}{n} \sum_{i=1}^{n} k_i(p)$$

or $\frac{1}{n}$ times the trace of L_p

Global Property exposes facts about the surface as a whole.

Local Property exposes facts about a surface at or near a particular point on the surface.

Global Theorem expresses hypothesis and conclusions about global properties. (Local Theorems refer to local properties.)

13 Convex Surfaces

Convex An oriented n-surface S in $\Re^{n+1}$is *convex*, or *globally convex* if, for each $p \in S$, S is contained in one of the closed half-spaces

$$H_p^+ = \left\{q \in \Re^{n+1} : (q-p) \cdot \mathbf{N}(p) \geq 0\right\}$$

or

$$H_p^- = \left\{q \in \Re^{n+1} : (q-p) \cdot \mathbf{N}(p) \leq 0\right\}$$

where $\mathbf{N}$is the Gauss Map of S.

Strictly Convex S is said to be *strictly convex* if S is convex and $S \bigcap H_p = p$ for each $p \in S$, where

$$H_p = q \in \Re^{n+1} : (q-p) \cdot \mathbf{N}(p) = 0$$

Gradient Vector Field of h is the smooth *tangent* vector field $\text{grad}\, h$ on S defined by

$$(\text{grad}\, h)\,(p) = \nabla \tilde{h}(p) - \left(\nabla \tilde{h}(p) \cdot \mathbf{N}(p)\right) \mathbf{N}(p)$$

where $\tilde{h}$ is any extension of h to a smooth function on an open set containing S, and $\mathbf{N}$is any orientation on S. Thus $\text{grad}\, h$ is the tangential component of $\nabla \tilde{h}$.

The gradient of $h : S \to \Re$ has the following properties:

(i) $\nabla_{\mathbf{v}} h = (\text{grad}\, h)\,(p) \cdot \mathbf{v} \;\; \forall\, \mathbf{v} \in S_p, p \in S$

(ii) $(h \circ \alpha\,)'(t) = (\text{grad}\, h)\,(\alpha\,(t)) \cdot \dot{\alpha}\,(t) \;\; \forall\, t \in I,$where $\alpha : I \to S$ is any parameterized curve in S.

(iii) $(\text{grad}\, h)\,(p) = \sum_{i=1}^{n} (\nabla_{\mathbf{v_i}} h)\, \mathbf{v_i}$, where $v_1, \ldots, v_n$ is any orthonormal basis for S_p, $p \in S$.

(iv) $(\text{grad}\, h)\,(p) = \mathbf{0}$ if and only if h is *stationary* at p, $p \in S$.

Stationary $h : S \to \Re$ is *stationary* at $p \in S$ if $\nabla_{\mathbf{v}} h = 0 \;\; \forall \mathbf{v} \in S_p$

Critical Point A point $p \in S$ at which $h : S \to \Re$ is *stationary* is called a *critical point* of h. Critical points occur where $h : S \to \Re$ at $p \in S$ attains:

(i) a *local minimum*, $h(q) \geq h(p) \;\; \forall\, q \in V$ ($V \subset S$ is an open set in S, $V = W \cap S$ for some open set $W \in \Re^{n+1}$).

(ii) a *local maximum*, $h(q) \leq h(p) \;\; \forall\, q \in V$.

(iii) a *saddle point*, h is stationary but is neither a local minimum or maximum

If the inequalities are strict, ($<$ or $>$) instead of ($\leq$ or $\geq$), then h is said to have attained a *strict* local minimum or maximum.

Hessian Let $p \in S$ be a critical point of $h : S \to \Re$. The *Hessian* of h at p is the quadratic form $\mathcal{H}_p : S_p \to \Re$ defined by

$$\mathcal{H}_p(\mathbf{v}) = \nabla_{\mathbf{v}} \,(\text{grad}\, h) \cdot \mathbf{v}$$

Thus $\mathcal{H}_p$is the quadratic form associated with the self-adjoint linear transformation on S_p which sends $\mathbf{v}$to $\nabla_{\mathbf{v}}\text{grad}\, h$. Note, $\nabla_{\mathbf{v}}\text{grad}\, h$ does lie in S_p $\forall \mathbf{v} \in S_p$.

Gradient Lines of h are the integral curves of grad h.

Non-Degenerate A critical point p of a smooth function $h : S \to \Re$ is *non-degenerate* if $\nabla_{\mathbf{v}}\,(\text{grad } h) \neq 0\ \forall \mathbf{v} \in S_p, \mathbf{v} \neq 0$. Non-degenerate critical points are isolated.

14 Parameterized Surfaces

Let U be an open set in $\Re^n$ and let $\phi : U \to \Re^m$ be a smooth map. *ϕ is a vector valued function.*

differential of ϕ is the smooth map $d\phi : U \times \Re^n \to \Re^m \times \Re^m$ defined as follows: A point $\mathbf{v} \in U \times \Re^n$ is a vector $\mathbf{v} = (p, v)$ at a point $p \in U$. Given $\mathbf{v}$, let $\alpha : I \to U$ be any parameterized curve in U with $\dot{\alpha}(t_o) = \mathbf{v}$. Then $d\phi(\mathbf{v})$ is the vector at $\phi(p)\ \left(d\phi\,(\mathbf{v}) \in \Re^m_{\phi(p)} \subset \Re^m \times \Re^m\right)$ defined by

$$d\phi\,(\mathbf{v}) = \phi\dot{\circ}\alpha(t_0)$$

The restriction $d\phi_p$ of $d\phi$ to $\Re^n_p$ is a linear map $d\phi_p : \Re^n_p \to \Re^m_{\phi(p)}$. Its matrix relative to the standard bases for $\Re^n_p$ and $\Re^m_{\phi(p)}$ is just the Jacobian matrix $((\partial\phi_i/\partial x_j)\,(p))$ of ϕ at p.

This second definition is related to the 1-form differential by

$$(d\phi)^{(2)}\,(t,u) = \left(\phi\,(t)\,, (d\phi)^{(1)}\,(t,u)\right)$$

Tangent Bundle The set $U \times \Re^n = \bigcup_{p\in U} \Re^n_p$ is called the *tangent bundle* of the open set U in $\Re^n$.

Regular means that $d\phi_p$ is non-singular (has rank n) for each $p \in U$

Parameterized n-surface in $\Re^{n+k}$ $(k \geq 0)$ is a smooth map $\phi : U \to \Re^{n+k}$, where U is a connected open set in $\Re^n$, which is *regular*.

The regularity condition gaurantees that the image of $d\phi_p$ is an n-dimensional subspace of $\Re^{n+k}_{\phi(p)}$. Image $d\phi_p$ is the *tangent space* to ϕ corresponding to the point $p \in U$. ϕ is not necessarily one-to-one.

Vector field along ϕ , where $\phi : U \to \Re^{n+k}$is a map $\mathbf{X}$which assigns to each point $p \in U$ a vector $\mathbf{X}(p) \in \Re^{n+k}_{\phi(p)}$. $\mathbf{X}$ is smooth if it is smooth as a map $\mathbf{X} : U \to \Re^{2(n+k)}$.

The vector field $\mathbf{X}$ is tangent to ϕ if it is of the form $\mathbf{X}(p) = d\phi_p\,(Y\,(p))\ \ (p \in U)$ for some vector field $\mathbf{Y}$ on U; $\mathbf{X}$ is normal to ϕ if $\mathbf{X}(p)\ \perp\ \text{Image}\, d\phi_p\ \ \forall p \in U$

Velocity Field , $\dot{\phi}$, is a tangent vector field along ϕ since $\dot{\phi}(t) = d\phi_t(t,1)\ \ \forall t$. The velocity field generalizes as follows: For $\phi : U \to \Re^{n+k}$a smooth map, U open in R^n, let $\mathbf{E}_i\ \ (i \in \{1,\ldots,n\})$ denote the tangent vector fields along ϕ defined by

$$\mathbf{E}_i(p) = d\phi_p\,(p, 0, \ldots, 1, \ldots, 0)$$

with the 1 is the i^{th} spot after the p. The components of $\mathbf{E}_i$ are the entries in the i^{th} column of the Jacobian matrix for ϕ at p:

$$\mathbf{E}_i(p) = \left(\phi\,(p)\,, \frac{\partial\phi}{\partial u_i}\,(p)\right) = \left(\phi\,(p)\,, \frac{\partial\phi_1}{\partial u_1}\,(p)\,, \ldots, \frac{\partial\phi_{n+k}}{\partial u_{n+k}}\,(p)\right),$$

Coordinate Vector Fields are the $\mathbf{E}_i$ defined above along ϕ. Each $\mathbf{E}_i$ is the velocity at p of the coordinate curve $u_i \to \phi(u_1, \ldots, u_n)$ (for all u_j held constant except u_i) passing through $\phi(p)$. When $\phi(p)$ is a parameterized n-surface (ϕ is regular), these vectors are linearly independent at each point $p \in U$ since $d\phi_p$ is non-singular, so they form a basis for the tangent space Image $d\phi_p$ for each $p \in U$.

Derivative along ϕ For $\phi : U \to \Re^{n+k}$a smooth map, U open in $\Re^n$, and $\mathbf{X}$ a smooth vector field along ϕ, the *derivative* $\nabla_{\mathbf{v}}\mathbf{X} \in \Re^{n+k}_{\phi(p)}$ of $\mathbf{X}$ with respect to $\mathbf{v} \in \Re^n_p$, $p \in U$, is defined by

$$\begin{aligned} \nabla_{\mathbf{v}}\mathbf{X} &= \left(\phi(p), \tfrac{d}{dt}\big|_{t_0}(X \circ \alpha)\right) \\ &= (\phi(p), \nabla_{\mathbf{v}}X_1, \ldots, \nabla_{\mathbf{v}}X_{n+k}) \end{aligned}$$

where X is the *vector part* of $\mathbf{X}$ ($\mathbf{X}(q) = (\phi(q), X(q))$ for all $q \in U$), α is any parameterized curve in U with $\dot{\alpha}(t_o) = \mathbf{v}$, and the $X_i : U \to \Re$ are the components of $\mathbf{X}$ ($\mathbf{X}(q) = (\phi(q), X_1(q), \ldots, X_{n+k}(q))$ for all $q \in U$)

If $\mathbf{v} = \mathbf{e_i} = (p, 0, \ldots, 1, \ldots, 0)$, then

$$\nabla_{\mathbf{e_i}}\mathbf{X} = \left(\phi(p), \frac{\partial X}{\partial u_i}(p)\right) = \left(\phi(p), \frac{\partial X_1}{\partial u_i}(p), \ldots, \frac{\partial X_{n+k}}{\partial u_i}(p)\right)$$

Orientation Vector Field along ϕ, denoted $\mathbf{N}$, is a vector field defined for $\phi : U \to \Re^{n+1}$, a parameterized $n-$surface in $\Re^{n+1}$, such that $\mathbf{N}(p)$ is a unique unit vector at $\phi(p)$ such that $\mathbf{N}(p) \perp$ Image $d\phi_p$ and

$$\det \begin{pmatrix} \mathbf{E}_1(p) \\ \cdots \\ \mathbf{E}_n(p) \\ \mathbf{N}(p) \end{pmatrix} > 0$$

where the function *det* is defined in Theorem 5 of Chapter 12. (The determinant of the matrix defined by the coefficients of each of the E_i and $\mathbf{N}$ vectors in row vector form.)

The Weingarten map is defined for this Orientation as

$$L_p : (\text{Image } d\phi_p) \to (\text{Image } d\phi_p)$$

defined by

$$L_p(d\phi_p(\mathbf{v})) = -\nabla_{\mathbf{v}}\mathbf{N}$$

The Weingarten map at $p \in U$ of the parameterized surface $\phi : U \to \Re^{n+1}$is a linear map, which is well defined since $d\phi_p$ is one-to-one. The relationship between eigenvalues/vectors of this map and principal curvatures, as defined before, holds for the parameterized $n-$surface .

For 2-surfaces,

$$\mathbf{N} = \frac{\mathbf{E_1} \times \mathbf{E_2}}{\|\mathbf{E_1} \times \mathbf{E_2}\|}$$

15 Local Equivalence of Surfaces and Parameterized Surfaces

Inverse Function Theorem *from Multi-variable Calculus* Let U be an open set in $\Re^{n+1}$, let $\psi : U \to \Re^{n+1}$ be smooth, and suppose $p \in U$ is such that $d\psi_p$ is non-singular. Then there exists an open set $V \subset U$ about p such that the restriction $\psi|_V$ of ψ to V maps V one-to-one onto an open set W in $\Re^{n+1}$, and moreover the inverse map $(\psi|_V)^{-1} : W \to V$ is smooth.

Note, since $d\psi_p$ with respect to the standard bases for $\Re^{n+1}_p$ and $\Re^{n+1}_{\psi(p)}$ is the Jacobian matrix $J_\psi(p)$ of ψ at p, the condition that $d\psi_p$ is non-singular says simply that $\det J_\psi(p) \neq 0$.

Local Parameterization A parameterized n-surface $\phi : U \to \Re^{n+1}$ whose image is an open subset of the oriented n-surface S and whose orientation vector field agrees with that of S (i.e.,$\mathbf{N}^\phi = \mathbf{N}^S \circ \phi$).

Chart The inverse ϕ^-1 of such a parameterization $\phi : U \to S$ is called a *chart*. This is because through ϕ^{-1} the region Image $\phi \subset S$ is "charted" on $U \subset \Re^n$.

Coordinate System , another name for "chart", because through ϕ^{-1} each point $p \in$ Image ϕ corresponds to an n-tuple of real numbers, the *coordinates* of p.

Atlas is a collection of charts which are compatible in the overlaps.

Stereographic Projection is a chart which maps $\mathbf{S}^n - \{q\}$ onto the equatorial hyperplane.

Surface of dimension n or n-surface in $\Re^{n+k}$ $(k \geq 1)$ is a non-empty subset S of $\Re^{n+k}$ of the form $S = f^{-1}(c)$ $(c \in \Re^k)$ where $f : U \to \Re^k$ (U open in $\Re^{n+k}$) is a smooth function with the property that df_p has rank k for each $p \in S$.

Since the matrix for df_p with respect to the standard bases for $\Re^{n+k}_p$ and $\Re^k_{\phi(p)}$ is just the Jacobian matrix of f, whose rows are the vector parts of the gradient vectors $\nabla f_i(p)$, where $f(q) = (f_1(q), \ldots, f_k(q))$, $q \in U$, this definition can be rephrased as follows:

an n-surface in $\Re^{n+k}$is a non-empty subset of $\Re^{n+k}$of the form

$$S = f_1^{-1}(c_1) \cap \cdots \cap f_k^{-1}(c_k) = \bigcap_{i=1}^{k} f_i^{-1}(c_i)$$

where $f_i : U \to \Re$ (U open in $\Re^{n+k}$ are smooth fucntions such that $\{\nabla f_1(p), \ldots, \nabla f_k(p)\}$ is linearly independent for each $p \in S$.

Thus an $n-$surface is the intersection of k $(n + k - 1)$-surfaces which meet "cleanly" in the sense that the normal directions are linearly independent at each point of intersection.

Tangent Space S_p at $p \in S$to an $n-$surface $S = \bigcap_{i=1}^k f_i^{-1}(c_i)$ in $\Re^{n+k}$is the set of all vectors in $\Re^{n+k}_p$ of the form $\dot{\alpha}(t_0)$ where α is any parameterized curve in S with $\alpha(t_0) = p$. Thus,

$$\begin{aligned} S_p = & \left[f_1^{-1}(c_1)\right]_p \cap \cdots \cap \left[f_k^{-1}(c_k)\right]_p \\ = & \left\{\mathbf{v} \in \Re^{n+k} : \nabla f_i(p) \cdot \mathbf{v} = 0 \text{ for } i \in \{1, \ldots, n\}\right\} \end{aligned}$$

Normal Space to S at p is the k-dimensional subspace $S_p^{\perp}$ of $\Re_p^{n+k}$ spanned by the vectors $\{\nabla f_1(p), \ldots, \nabla f_k(p)\}$.

Space Curve A 1-surface in $\Re^3$

Smoothness For $f : S \to \Re^k$ a function defined on an n-surface S in $\Re^{n+k}$, the smoothness of f can be equivalently defined by:

(i) the requirement that f be the restriction to S of a smooth function defined on some open set in $\Re^{n+k}$containing S, or

(ii) by the requirement that $f \circ \phi$ be smooth for each local parameterization ϕ of S.

Diffeomorphism a smooth map with a smooth inverse.

16 Focal Points

Later *this chapter will be done later*

17 Surface Area and Volume

Volume of a parameterized $n-$surface $\phi : U \to \Re^{n+1}$is defined to be

$$V(\phi) = \int_U \det \begin{pmatrix} \mathbf{E}_1 \\ \vdots \\ \mathbf{E}_n \\ \mathbf{N} \end{pmatrix} = \int_U \det \begin{pmatrix} \mathbf{E}_1(u_1, \ldots, u_n) \\ \vdots \\ \mathbf{E}_n(u_1, \ldots, u_n) \\ \mathbf{N}(u_1, \ldots, u_n) \end{pmatrix} du_1 \ldots du_2$$

or,

$$V(\phi) = \int_U \det \begin{pmatrix} \mathbf{E}_1 \cdot \mathbf{E}_1 & \cdots & \mathbf{E}_1 \cdot \mathbf{E}_n & 0 \\ \vdots & & \vdots & \vdots \\ \mathbf{E}_n \cdot \mathbf{E}_1 & \cdots & \mathbf{E}_n \cdot \mathbf{E}_n & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} = \int_U (\mathbf{E}_i \cdot \mathbf{E}_j)^{\frac{1}{2}}$$

where $\mathbf{E}_1, \ldots, \mathbf{E}_n$ are the coordinate vector fields along ϕ and $\mathbf{N}$is the orientation vector field along ϕ.

If $n = 1$, the volume of ϕ is usually called *length.*

If $n = 2$, the volume of ϕ is usually called *area.*

Metric Coefficients along ϕ are the functions $g_{ij} = \mathbf{E}_i \cdot \mathbf{E}_j$ used in the formula above. The determinant of the matrix of these functions, $\det(g_{ij})$ is normally denoted by the letter g in differential geometry literature.

Singular adjective used to describe an n-surface which is not *regular.* (i.e., *singular n-surface*)

Standard Orientation ???

Orientation Preserving A smooth map $f : S \to \tilde{S}$ is *orientation preserving* if *df* maps bases consistent with the orientation on S to bases for $\tilde{S}_{f(p)}$ consistent with the orientation on $\tilde{S}$. Otherwise, the map is *orientation reversing.*

Reparameterization of a singular n-surface $\phi : U_1 \to \Re^{n+k}$ is a singular n-surface $\psi : U_2 \to \Re^{n+k}$ of the form $\psi = \phi \circ h$ where $h : U_2 \to U_1$ is a smooth map with a smooth inverse and a Jacobian determinant everywhere positive.

Volume Form on S, denoted ζ, is the function which assigns to each ordered set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of n vectors in S_p the number

$$\zeta(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = \det \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \\ \mathbf{N}(p) \end{pmatrix}$$

The integral of the volume form, integrated over the surface S, is the *volume* of S.

Differential k-form , or simply a k-form, on an $n-$surface S is a function ω which assigns to each ordered set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of k vectors in S_p, $p \in S$, a real number $\omega(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ such that

1. *(Multilinearity):* for each $i \in \{1, \ldots, k\}$ and $\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_k \in S_p$, $p \in S$, the function which sends $\mathbf{v} \in S_p$ to $\omega(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_k) \in \Re$ is linear, and
2. *(Skewsymmetry):* for each $\mathbf{v}_1, \ldots, \mathbf{v}_k \in S_p$ and for each permutation σ of the integers $\{1, \ldots, k\}$,

$$\omega\left(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(k)}\right) = (\mathrm{sign}\sigma)\, \omega(\mathbf{v}_1, \ldots, \mathbf{v}_k).$$

Property 1 means that the k-form is linear for each variable with the other variables fixed. If $k = n$ and $\omega = \zeta$, then the k-form has the familiar properties of the determinant.

Exterior Product let ω_1 and ω_2 be 1-forms on S. Then the function $\omega_1 \wedge \omega_2$ defined by

$$(\omega_1 \wedge \omega_2)(\mathbf{v}_1, \mathbf{v}_2) = \omega_1(\mathbf{v}_1)\, \omega_2(\mathbf{v}_2) - \omega_1(\mathbf{v}_2)\, \omega_2(\mathbf{v}_1)$$

is a 2-form on S, called the *exterior product* of ω_1 and ω_2.

Interior Product Let ω be a k-form $(k > 1)$ on S and let $\mathbf{X}$ be a tangent vector field on S. Then the function $\mathbf{X} \lfloor\_\, \omega$ defined by

$$(\mathbf{X} \lfloor\_\, \omega)(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}) = \omega(\mathbf{X}(p), \mathbf{v}_1, \ldots, \mathbf{v}_{k-1})$$

is the $(k-1)$-form on S. $\mathbf{X} \lfloor\_\, \omega$ is the *interior product* of $\mathbf{X}$and ω.

Note: This is the "lower left bracket" thingy in Thorpe's notation. I can't figure out how to do that in LaTex, so I'm using $\lfloor\_$ *for now!*

Pullback Let ω be a k-form on an m-surface $\tilde{S}$ and let $f : S \to \tilde{S}$ be a smooth map from the n-surface S into $\tilde{S}$. The the function $f^*\omega$ defined by

$$(f^*\omega)(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \omega(df(\mathbf{v}_1), \ldots, df(\mathbf{v}_k))$$

is a k-form on S. $f^*\omega$ is called the *pullback* of ωunder f. { *The pullback defines the action of* ω *on* S.}

Sum of two k-forms ω_1 and ω_2 on an n-surface S is the k-form $\omega_1 + \omega_2$ defined by

$$(\omega_1 + \omega_2)(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \omega_1(\mathbf{v}_1, \ldots, \mathbf{v}_k) + \omega_2(\mathbf{v}_1, \ldots, \mathbf{v}_k)$$

Product of a function $f : S \to \Re$ and a k-form ω on S is the k-form $f\omega$ on S defined by

$$(f\omega)(\mathbf{v}_1, \ldots, \mathbf{v}_k) = f(p)\,\omega(\mathbf{v}_1, \ldots, \mathbf{v}_k)$$

Integral For a smooth k-form on an n-surface S and ϕ a singular k-surface in S (i.e.,, with Image $\phi \subset S$), the *integral of* ω *over* ϕ is defined by

$$\int_\phi \omega = \int_U \omega\left(\mathbf{E}_1^\phi, \ldots, \mathbf{E}_k^\phi\right)$$

provided the latter integral exists. The vector fields $\mathbf{E}_i^\phi$ are the coordinate vector fields along ϕ. The integral will exist whenever the function $\omega\left(\mathbf{E}_1^\phi, \ldots, \mathbf{E}_k^\phi\right)$ is identically zero outside some compact subset of U.

Volume of a compact oriented n-surface $S \subset \Re^{n+1}$ is the integral over S of its volume form ζ:

$$V(S) = \int_S \zeta$$

The integral over S of any smooth function $f : S \to \Re$ is given by

$$\int_S f = \int_S f\zeta$$

18 Minimal Surfaces

Variation Let $\phi : U \to \Re^{n+1}$be a parameterized $n-$surface in $\Re^{n+1}$. A variation of ϕ is a smooth map $\psi : U \times (-\epsilon, \epsilon) \to \Re^{n+1}$ with the property that $\psi(p, 0) = \phi(p)\ \forall\, p \in U$. Thus a variation surrounds the $n-$surface ϕ with a family of n-surfaces $\phi_s : U \to \Re^{n+1}\ (-\epsilon < s < \epsilon)\, z$ defined by $\phi_s(p) = \psi(p, s)$.

Normal Variation is a variation of the form

$$\psi(p, s) = \phi(p) + sf(p)\mathbf{N}(p)$$