Q: How do mathematicians induce good behavior in their children?

A: 'If I've told you n times, I've told you n+1 times...'

CS151 Fall 2014
Lecture 9 – 9/23
Induction

Prof. Tanya Berger-Wolf
http://www.cs.uic.edu/CS151

Adapted from Lap Chi Lau – The Chinese University of Hong Kong and David Liben-Nowell – Carleton College

Announcements

- **Homework 3 is out. Due in a week Sept 30 in class**
- **Microsoft event: Meet the Company (9/24/14)**
  Find out what Microsoft is up to this year and what a day in the life is like straight from full time engineers who will share on the technologies they work on every day and tips on how to land yourself a job here. Don't forget your resume! **Free food and swag!**
  Where: Student Center East (SCE): Illinois Room- C
  When: Wednesday, September 24, 6:00 pm
- **Microsoft event: College Code Competition (9/30/14)**
  Are you the best coder on campus? Prove it. Team up with up to 2 friends. **Bring your laptop and your Microsoft Live ID.**

  You could win $100 or a Dell Venue Pro. And yes, we'll feed you.

  Where: Student Center East (SCE): Illinois Room- AB
  When: Tuesday, September 30, 6:00 pm - 8:30 pm

Proving an Equality

$$\forall n \geq 1 \quad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Base case: P(1) is true

Let P(n-1) be the induction hypothesis that the statement is true for n-1.

Induction step: assume P(n-1) is true, prove P(n) is true.

$$\sum_{i=1}^{n} i^3 = \sum_{i=1}^{n-1} i^3 + n^3$$

$$= \left(\frac{(n-1)n}{2}\right)^2 + n^3 \qquad \text{by induction}$$

$$= \frac{(n-1)^2n^2 + 4n^3}{4} = \frac{n^2(n^2-2n+1+4n)}{4} = \frac{n^2(n+1)^2}{4}$$

$$= \left(\frac{n(n+1)}{2}\right)^2$$

Proving a Property

$$\forall n \geq 1, \quad 2^{2n} - 1 \text{ is divisible by } 3$$

Base Case ($n = 1$): $2^{2n} - 1 = 2^2 - 1 = 3$

Induction Step: Assume $P(n-1)$ for some $n \geq 1$ and prove $P(n)$:

Assume $2^{2(n-1)} - 1$ is divisible by 3, prove $2^{2n} - 1$ is divisible by 3.

$$2^{2n} - 1 = 2^{2(n-1)+2} - 1$$

$$= 4 \cdot 2^{2(n-1)} - 1$$

$$= \underbrace{3 \cdot 2^{2(n-1)}}_{\text{Divisible by 3}} + \underbrace{2^{2(n-1)} - 1}_{\text{Divisible by 3 by induction}}$$

Paradox?

Theorem: All horses are the same color.

Theorem: Let P(**n**) be "for all sets S of size n of horses, all horses in S have the same color".
Then **for all n >= 0, P(n).**

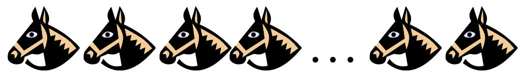
Proof: (by induction on *n*)

Base case (*n*=0):

No horses so *vacuously* true!

Induction hypothesis:

P(*n-1*) ::= for all sets S of size n-1 of horses, all horses in S have the same color

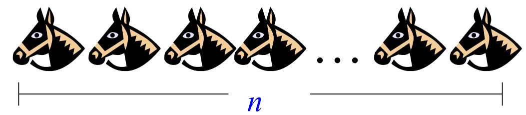


Paradox

(Inductive case)

Assume any *n-1* horses have the same color.

Prove that any *n* horses have the same color.



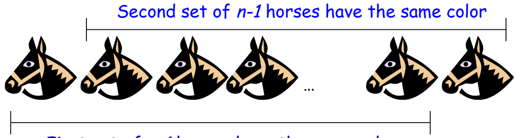
Paradox

(Inductive case)

Assume any *n-1* horses have the same color.

Prove that any *n* horses have the same color.

Second set of *n-1* horses have the same color



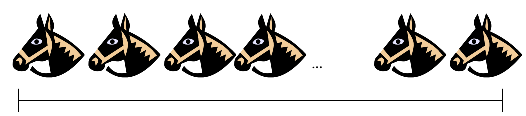
First set of *n-1* horses have the same color

Paradox

(Inductive case)

Assume any *n-1* horses have the same color.

Prove that any *n* horses have the same color.

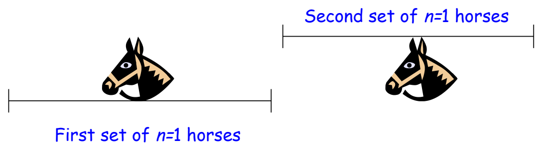


Therefore the set of *n* have the same color!

Paradox

What is wrong? $n = 2$

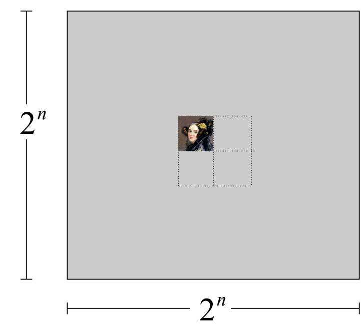
Proof that $P(n\text{-}1) \rightarrow P(n)$ is false if $n = 2$, because the two horse groups *do not overlap*.



(But proof works for all $n \neq 2$)

Puzzle

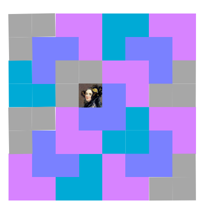
Goal: tile the squares, except one in the middle for Ada.



Puzzle

There are only L-shaped tiles covering three squares:

For example, for 8 x 8 puzzle might tile for Ada this way:



Puzzle

Theorem: For any $2^n \times 2^n$ puzzle, there is a tiling with Ada in the middle.

Did you remember that we proved $2^{2n} - 1$ is divisible by 3?

Proof: (by induction on n)

$P(n)$ ::= can tile $2^n \times 2^n$ with Ada in middle.

Base case: (n=0)

(no tiles needed)