CS151 Fall 2014
Lecture 17 – 10/23
Functions

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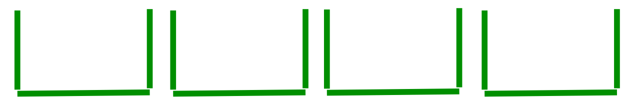
Pigeonhole Principle

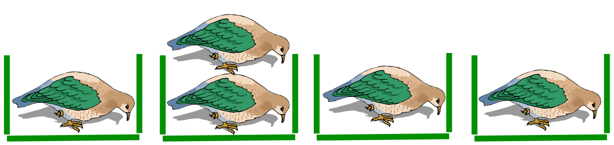
If more pigeons

than pigeonholes,



Pigeonhole Principle

then some hole must have at least two pigeons!



Pigeonhole principle

A function from a larger set to a smaller set cannot be injective.

(There must be at least two elements in the domain that have the same image in the codomain.)

Example

Question: Let A = {1,2,3,4,5,6,7,8}

If five integers are selected from A, must a pair of integers have a sum of 9?

Consider the pairs {1,8}, {2,7}, {3,6}, {4,5}.
Among them, they contain all the numbers in A
The sum of each pair is equal to 9.

The pairs are the holes.

If we choose 5 numbers from the set A,

(those numbers are the pigeons)

then by the pigeonhole principle,

both elements of some pair will be chosen,

and their sum is equal to 9.

Birthday Paradox

In a group of 366 people, there must be two people having the same birthday.

Suppose n <= 365, what is the probability that in a random set of n people, some pair of them will have the same birthday?

We can think of it as picking n random numbers from 1 to 365 without repetition.

There are 365^n ways of picking n numbers from 1 to 365.

There are 365·364·363·…·(365-n+1) ways of picking n numbers from 1 to 365 without repetition.

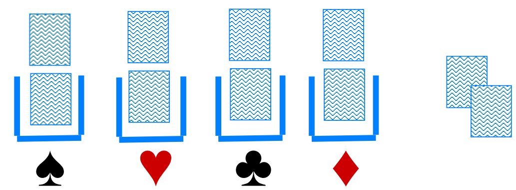
So the probability that no pairs have the same birthday is equal to 365·364·363·…·(365-n+1) / 365^n

This is smaller than 50% for 23 people, smaller than 1% for 57 people.

Generalized Pigeonhole Principle

Generalized Pigeonhole Principle

If n pigeons and h holes, then some hole has at least $\left\lceil \frac{n}{h} \right\rceil$ pigeons.



Cannot have < 3 cards in every hole.

Subset Sum

204801353855029644448038	3171004832173501394113017	5763257331083479647409398	8247331000042995311646021
489445991866915676240992	3208234421597368647019265	5800949123548989122628663	8496243997123475922766310
1082662032430379651370981	3437254656355157864869113	6042900801199280218026001	8518399140676002660747477
1178480894769706178994993	3574883393058653923711365	6116171789137737896701405	8543691283470191452333763
1253127351683239693851327	3644909946040480189969149	6144868973001582369723512	8675309258374137092461352
1301505129234077811069011	3790044132737084094417246	6247314593851169234746152	8694321112363996867296665
1311567111143866433882194	3870332127437971355322815	6814428944266874963488274	8772321203608477245851154
1470029452721203587686214	4080505804577801451363100	6870852945543886849147881	8791422161722582546341091
1578271047286257499433886	4167283461025702348124920	6914955508120950093732397	9062628024592126283973285
1638243921852176243192354	4235996831123777788211249	6949632451365987152423541	9137845566925526349897794
1763580219131985963102365	4670939445749439042111220	7128211143613619828415650	9153762966803189291934419
1826227795601842231029694	4815379351865384279613427	7173920083651862307925394	9270880194077636406984249
1843971862675102037201420	4837052948212922604442190	7215654874211755676220587	9324301480722103490379204
2396951193722134526177237	5106389423855018550671530	7256932847164391040233050	9436090832146695147140581
2781394568268599801096354	5142368192004769218069910	7332822657075235431620317	9475308159734538249013238
2796605196713610405408019	5181234096130144084041856	7426441829541573444964139	9492376623917486974923202
2931016394761975263190347	5198267398125617994391348	7632198126531809327186321	9511972558779880288252979
2933458058294405155197296	5317592940316231219758372	7712154432211912882310511	9602413424619187112552264
3075514410490975920315348	5384358126771794128356947	7858918664240262356610010	9631217114906129219461111
3111474985252793452860017	5439211712248901995423441	7898156786763212963178679	9908189853102753335981319
3145621587936120118438701	5610379826092838192760458	8147591017037573337848616	9913237476341764299813987
3148901255628881103198549	5632317555465228677676044	8149436716871371161932035	
3157693105325111284321993	5692168374637019617423712	8176063831682536571306791	

Two different subsets of the 90 25-digit numbers shown above have the same sum.

Subset Sum

90 numbers, each with at most 25 digits.

So the total sum is at most 90×10^{25}

Let A be the set of all subsets of the 90 numbers. (pigeons)

Let B be the set of integers from 0 to 90×10^{25}. (pigeonholes)

$$|A| = 2^{90} \geq 1.237 \times 10^{27}$$

$$|B| = 90 \times 10^{25} + 1 \leq 0.901 \times 10^{27}$$

By pigeonhole principle, there are two different subsets with the same sum.

This is an example of a **non-constructive** proof.

Cardinality

Functions are useful to compare the sizes of two different sets.

Question: Are all infinite sets having the same cardinality?

Two sets A and B have the same cardinality if and only if there is a bijection between A and B.

A set is countable if it has the same cardinality as the set of positive integers.

Integers vs Positive Integers

Is the set of integers countable?

Define a bijection between the positive integers and all integers

```
1 2  3 4  5 6  7 8...
0 1 -1 2 -2 3 -3 4...
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$$f(n) = \begin{cases} n/2, & \text{if } n \text{ is even;} \\ -(n-1)/2, & \text{if } n \text{ is odd.} \end{cases}$$

So, the set of integers is countable.

Rational Numbers vs Positive Integers

Question: Is the set of **rational number** countable?

The set of "pair of integers" (a,b) is not smaller than the set of rational number.

We want to show that the set of "pair of integers" is countable, by defining a **bijection** to the set of positive integers.

This would then imply the set of rational is countable.

Rational Numbers vs Positive Integers

```
(0, 0), (0, 1), (0,-1), (0, 2), (0,-2), (0, 3), (0,-3), . . .
(1, 0), (1, 1), (1,-1), (1, 2), (1,-2), (1, 3), (1,-3), . . .
(-1, 0),(-1, 1),(-1,-1),(-1, 2),(-1,-2),(-1, 3), (-1,-3), . . .
(2, 0), (2, 1), (2,-1), (2, 2), (2,-2), (2, 3),  (2,-3), . . .
(-2, 0),(-2, 1),(-2,-1),(-2, 2),(-2,-2), (-2, 3),(-2,-3), . . .
```

If you map the set of positive integers to the top row first, then you will not be able to reach the second row.

The trick is to visit the rational numbers diagonal by diagonal.

Each diagonal is finite, so eventually every pair will be visited.

Therefore, there is a **bijection** from the set of positive integers, to the set of pair of integers, and so the set of rational numbers is countable.