Real Numbers vs Positive Integers

Question: Is the set of **real number** countable?

Theorem: **No surjection** mapping positive integers to real numbers.

Theorem: **No surjection** mapping positive integers to reals between 0 and 1.

The string map to the first natural number → $f(1) = 0.d_{11}d_{12}d_{13}d_{14}d_{15}d_{16}d_{17}d_{18}\ldots$

$f(2) = 0.d_{21}d_{22}d_{23}d_{24}d_{25}d_{26}d_{27}d_{28}\ldots$

$f(3) = 0.d_{31}d_{32}d_{33}d_{34}d_{35}d_{36}d_{37}d_{38}\ldots$

$f(4) = 0.d_{41}d_{42}d_{43}d_{44}d_{45}d_{46}d_{47}d_{48}\ldots$

The string map to the fifth natural number → $f(5) = 0.d_{51}d_{52}d_{53}d_{54}d_{55}d_{56}d_{57}d_{58}\ldots$

$f(6) = 0.d_{61}d_{62}d_{63}d_{64}d_{65}d_{66}d_{67}d_{68}\ldots$

$f(7) = 0.d_{71}d_{72}d_{73}d_{74}d_{75}d_{76}d_{77}d_{78}\ldots$

$f(8) = 0.d_{81}d_{82}d_{83}d_{84}d_{85}d_{86}d_{87}d_{88}\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

It can not be in any row i because its i-th digit is different, and so this string is not mapped!

The opposite of the diagonal → $r = 0.r_1r_2r_3r_4r_5r_6r_7r_8\ldots$

$$r_i = \begin{cases} 1 & \text{if } d_{ii} \neq 1 \\ 2 & \text{if } d_{ii} = 1 \end{cases}$$

Diagonal Argument

Similarly, power sets can be shown to be **uncountable**.

This argument is called Cantor's diagonal argument.

http://en.wikipedia.org/wiki/Cantor's_diagonal_argument

	0	1	2	3	4	
$f(0)$	1	0	1	0	1	⋯
$f(1)$	0	0	0	1	1	⋯
$f(2)$	0	1	1	0	1	⋯
$f(3)$	1	1	0	1	1	⋯
$f(4)$	1	0	1	0	0	⋯
	⋮	⋮	⋮	⋮	⋮	⋱

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This has been used in many places; for example the Russell's paradox.

$$T = \{\, s \in S : s \notin f(s) \,\}.$$

Cardinality and Computability

The set of all computer programs in a given computer language is countable.

The set of all functions is uncountable.

There must exist a non-computable function!