CS15 Fall 2014
Lecture 26 – 11/25

Relations

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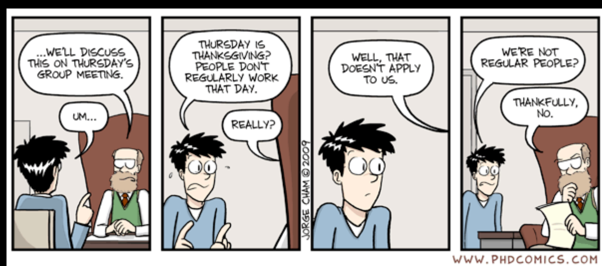
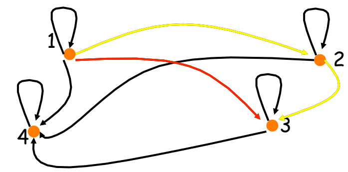


Properties of Relations - techniques…

(a,b) ∈ R and (b,c) ∈ R implies (a,c) ∈ R.

How can we check for **transitivity**?
Draw a picture of the relation (called a "graph").
- **Vertex/node** for every element of *A*
- **Edge/link** for every element of R

R={(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}



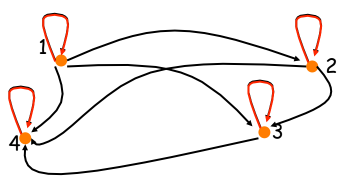
A "short cut" must be present for EVERY path of length 2.

Properties of Relations - techniques…

How can we check for the **reflexive property**?
Draw a picture of the relation (called a "graph").
- Vertex for every element of *A*
- Edge for every element of R

(a,a) ∈R.

R={(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}



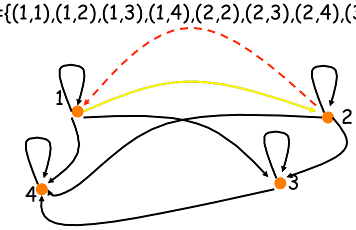
Loops must exist on EVERY vertex.

Properties of Relations - techniques…

How can we check for the symmetric property?
Draw a picture of the relation (called a "graph").
- Vertex for every element of *A*
- Edge for every element of R

(a,b) ∈ R implies (b,a) ∈ R

R={(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}



EVERY edge must have a return edge.

Properties of Relations - techniques…

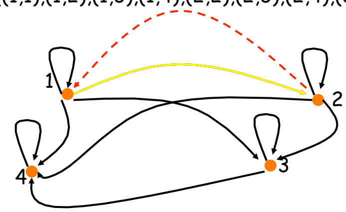
How can we check for the **anti-symmetric property**?

$(a,b) \in R$ implies $(b,a) \notin R$.

Draw a picture of the relation (called a "graph").

- Vertex for every element of A
- Edge for every element of R

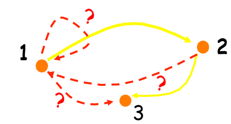
R={(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}



NO edge can have a return edge.

Properties of Relations - techniques…

Let R be a relation on People,
R={(x,y): x and y have lived in the same country}



Is R transitive? No

Is it symmetric? Yes

Is it reflexive? Yes

Is it anti-symmetric? No

More than one relation

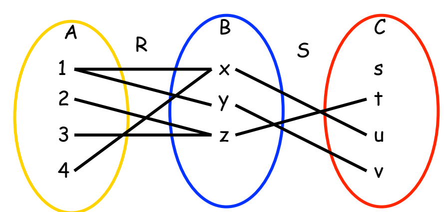
Suppose we have 2 relations, R_1 and R_2, and recall that relations are just sets! So we can take unions, intersections, complements, symmetric differences, etc.

There are other things we can do as well…

More than one relation

Let R be a relation from A to B ($R \subseteq A \times B$), and let S be a relation from B to C ($S \subseteq B \times C$). The composition of R and S is the relation from A to C ($S \circ R \subseteq A \times C$):

$S \circ R = \{(a,c): \exists b \in B, (a,b) \in R, (b,c) \in S\}$



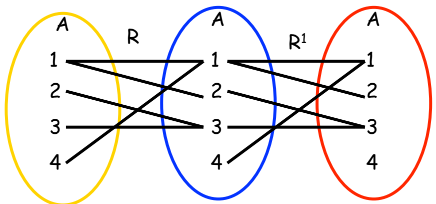
$S \circ R = \{(1,u),(1,v),(2,t),(3,t),(4,u)\}$

More than one relation

Let R be a relation on A. Inductively define

$R^1 = R$

$R^n = R^{n-1} \circ R$



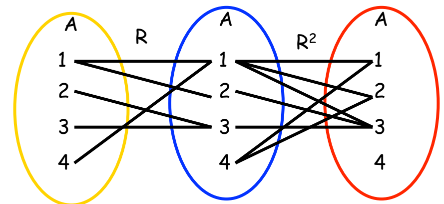
$R^2 = R^1 \circ R = \{(1,1),(1,2),(1,3),(2,3),(3,3),(4,1),(4,2)\}$

More than one relation

Let R be a relation on A. Inductively define

$R^1 = R$

$R^n = R^{n-1} \circ R$



$R^3 = R^2 \circ R = \{(1,1),(1,2),(1,3),(2,3),(3,3),(4,1),(4,2),(4,3)\}$

… = R4

= R5

= R6…

Relations - A Theorem:

If R is a transitive relation, then $R^n \subseteq R$, $\forall n$.

Proof by induction on n.

Base case (n=1): $R^1 \subseteq R$ because by definition, $R^1 = R$.

Inductive Step: IH: if R is transitive, then $R^{n-1} \subseteq R$.

Prove: if R is transitive, then $R^n \subseteq R$.

We are trying to prove that $R^n \subseteq R$. To do this, we select an element of R^n and show that it is also an element of R.

Typical way of proving subset.

Let (a,b) be an element of R^n. Since $R^n = R^{n-1} \circ R$, we know there is an x so that $(a,x) \in R$ and $(x,b) \in R^{n-1}$.

By IH, since $R^{n-1} \subseteq R$, $(x,b) \in R$.

But wait, if $(a,x) \in R$, and $(x,b) \in R$, and R is transitive, then $(a,b) \in R$.

Relations - A Theorem:

If R is a **transitive** relation, then $R^n \subseteq R$, $\forall n$.

***Aside*:** notice that this theorem allows us to conclude that the previous relation was NOT transitive.

Recall: "if p then q" ≡ "if not q then not p."

We saw that R^n was not a subset of R (it was growing on every iteration).

Therefore, R is not transitive.

Relations - more techniques…

Suppose we have our old relation R on AxB, where
A={1,2,3,4}, and B={u,v,w},
R={(1,u),(1,v),(2,w),(3,w),(4,u)}.

Then we can represent R as:

	u	v	w
1	1	1	0
2	0	0	1
3	0	0	1
4	1	0	0

The labels on the outside are for clarity. It's really the matrix in the middle that's important.

This is a |A| x |B| matrix whose entries indicate membership in R.

Relations - more techniques…

Some things to think about.
Let R be a relation on a set A, and let M_R be the matrix representation of R.
Then R is reflexive if, _______________.

	u	v	w
u	1	1	0
v	0	1	1
w	0	0	1

A. All entries in M_R are 1.
B. The \ diagonal of M_R contains only 1s.
C. The first column of M_R contains no 0s.
D. None of the above.

Relations - more techniques…

Some things to think about.
Let R be a relation on a set A, and let M_R be the matrix representation of R.
Then R is symmetric if, _______________.

	u	v	w
u	1	0	1
v	0	0	1
w	1	1	0

A. All entries above the \ are 1.
B. The first and last columns of M_R contain an equal # of 0s.
C. M_R is visually symmetric about the \ diagonal.
D. None of the above.

Relations - more techniques…

Suppose we have R1 and R2 defined on A:

R_1	u	v	w
u	1	0	1
v	0	0	1
w	1	1	0

R_2	u	v	w
u	1	1	0
v	0	1	1
w	0	0	1

Then $R_1 \cup R_2$ is the bitwise "or" of the entries: $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$M_{R1 \cup R2} = M_{R1} \vee M_{R2}$

Then $R_1 \cap R_2$ is the bitwise "and" of the entries: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$M_{R1 \cap R2} = M_{R1} \wedge M_{R2}$