CS15 Fall 2014
Lecture 27 – 12/2

Relations

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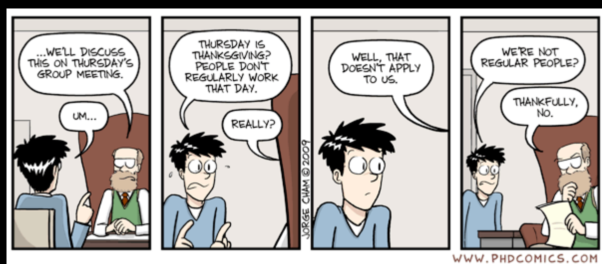


Partially Ordered Sets (POSets)

Let R be a relation then R is a Partially Ordered Set (POSet) if it is
- Reflexive - aRa, ∀a
- Transitive - aRb ∧ bRc → aRc, ∀a,b,c
- *Anti*symmetric - aRb ∧ bRa → a=b, ∀a,b

Ex. (**R**,≤), the relation "≤" on the real numbers, is a partial order.

How do you check?

= Reflexive? a ≤ a for any real

order Transitive? If a ≤ b, b ≤ c then a ≤ c

Actually total order. < Antisymmetric? If a ≤ b, b ≤ a then a = b

Partially Ordered Sets (POSets)

Ex. (Z^+, |), the relation "divides" on positive integers.

Reflexive? Yes, x|x since x=1x (k=1)

Transitive? Yes, or No?

a|b means b=ak, b|c means c=bj.
Does c=am for some m?

c = bj = akj (m=kj)

Antisymmetric?

a|b means b=ak, b|a means a=bj.
But b = bjk (subst) only if jk=1.

jk=1 means j=k=1, and we have b=a1, or b=a

Partially Ordered Sets (POSets)

Ex. (Z, |), the relation "divides" on integers. Not a poset.

Reflexive? Yes, x|x since x=1x (k=1)

Transitive?

a|b means b=ak, b|c means c=bj.
Does c=am for some m?

c = bj = akj (m=kj)

Antisymmetric? 3|-3, and -3|3, but 3 ≠ -3.

Yes, or No?

Partially Ordered Sets (POSets)

Ex. $(2^S, \subseteq)$, the relation "subset" on set of all subsets of S.

A poset.

Reflexive?

Yes, $A \subseteq A, \forall A \in 2^S$

Transitive?

$A \subseteq B, B \subseteq C$. Does that mean $A \subseteq C$?

$A \subseteq B$ means $x \in A \rightarrow x \in B$

$B \subseteq C$ means $x \in B \rightarrow x \in C$

Now take an x, and suppose it's in A. Must it also be in C?

Yes, by MP

A. Modus Ponens
B. Modus Tollens
C. DeMorgan's
D. Transitivity

$A \subseteq B, B \subseteq A \rightarrow A=B$

Partially Ordered Sets (POSets)

When we don't have a special relation definition in mind, we use the symbol "≤" to denote a partial order on a set.

When we know we're talking about a partial order, we'll write "a ≤ b" instead of "aRb" when discussing members of the relation.

We will also write "a < b" if a ≤ b and a ≠ b.

Partially Ordered Sets (POSets)

Ex. A common partial order on bit strings of length n, $\{0,1\}^n$, is defined as:

$a_1a_2\ldots a_n \leq b_1b_2\ldots b_n$

If and only if $a_i \leq b_i$, ∀ i.

A. 0110 ≤ 1000
B. 0110 ≤ 0000
C. 0110 ≤ 1110
D. 0110 ≤ 10111

0110 and 1000 are "incomparable" … We can't tell which is "bigger."

Actually, this relation is exactly the same as the last example, $(2^S, \subseteq)$.

In the string relation, we said 00≤01 because every bit in 00 is less than or = the corresp bit in 01.

String on the right has at least all the 1 bits of the left, maybe more. If each 1 represents an element in S, then right side has all elts of the left, maybe more.

Partially Ordered Sets (POSets)

Let (S, ≤) be a PO. If a ≤ b, or b ≤ a, then a and b are *comparable*. Otherwise, they are *incomparable*.

Ex. In poset $(Z^+, |)$, 3 and 6 are comparable, 6 and 3 are comparable, 3 and 5 are not, 8 and 12 are not.

A *total order* is a partial order where every pair of elements is comparable.

Ex. $(Z^+, \leq)$, is a total order, because for every pair (a,b) in ZxZ, either a ≤ b, or b ≤ a.

Dictionary order, or alphabetic order, or lexicographic order is a partial order on words in the english language. This idea can be generalized to strings over any alphabet.

Hasse Diagrams

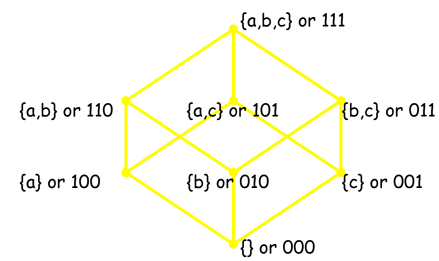
Hasse diagrams are a special kind of graphs used to describe posets.

Ex. In poset ({1,2,3,4}, ≤), we can draw the following picture to describe the relation.

1. Draw edge (a,b) if a ≤b
2. Don't draw up arrows
3. Don't draw self loops
4. Don't draw transitive edges

Hasse Diagrams

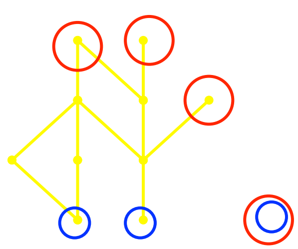
Have you seen this one before?



Hasse Diagrams

Consider this poset:

Reds are *maximal*.
Blues are *minimal*.

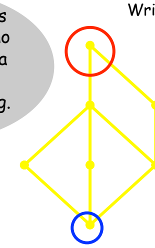


Hasse Diagrams

Definition: In a poset S, an element z is a *minimum element* if ∀b∈S, z≤b.

Intuition: If a is maxiMAL, then no one beats a. If a is maxiMUM, a beats everything.

Write the defn of maximum!



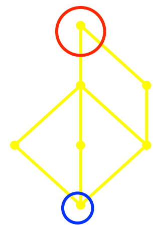
Did you get it right?

Must minimum and maximum exist?

A. Only if set is finite.
B. No.
C. Only if set is transitive.
D. Yes.

Hasse Diagrams

Theorem: In every poset, if the maximum element exists, it is unique. Similarly for minimum.



Proof: Suppose there are two maximum elements, a_1 and a_2, with $a_1 \neq a_2$.

Then $a_1 \leq a_2$, and $a_2 \leq a_1$, by defn of maximum.

So $a_1 = a_2$, a contradiction.

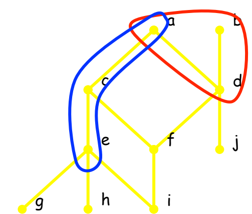
Thus, our supposition was incorrect, and the maximum element, if it exists, is unique.

Similar proof for minimum.

Upper and Lower Bounds

Defn: Let (S, ≤) be a partial order. If A⊆S, then an *upper bound* for A is any element x ∈ S (perhaps in A also) such that ∀ a ∈ A, a ≤ x.

A *lower bound* for A is any x ∈ S such that ∀ a ∈ A, a ≥ x.



Ex. The upper bound of {g,j} is a. Why not b?

Upper and Lower Bounds

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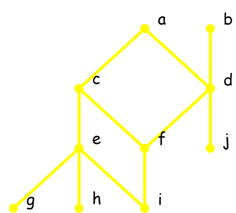


Ex. The upper bound of {g,j} is a. Why not b?

Ex. The upper bounds of {g,i} is/are…

A. I have no clue.

B. c and e

C. a

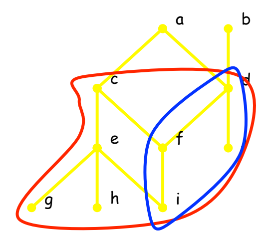
D. a, c, and

{a, b} has no UB.

Upper and Lower Bounds

Defn: Let (S, ≤) be a partial order. If A⊆S, then an *upper bound* for A is any element x ∈ S (perhaps in A also) such that ∀ a ∈ A, a ≤ x.

A *lower bound* for A is any x ∈ S such that ∀ a ∈ A, a ≥ x.



Ex. The lower bounds of {a,b} are d, f, i, and j.

Ex. The lower bounds of {c,d} is/are…

A. I have no clue.

B. f, i

C. j, i, g, h

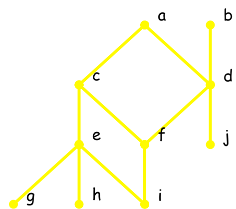
D. e, f, j

{g, h, i, j} has no LB.

Upper and Lower Bounds

Defn: Given poset (S, ≤) and A⊆S, x ∈ S is a *least upper bound* (LUB) for A if x is an upper bound and for upper bound y of A, y ≥ x.

x is a *greatest lower bound* for A if x is a lower bound and if x ≤ y for every UB y of A.



Ex. LUB of {i,j} = d.

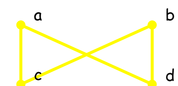
Ex. GLB of {g,j} is…
A. I have no clue.
B. a
C. non-existent
D. e, f, j

Upper and Lower Bounds

Ex. In the following poset, c and d are lower bounds for {a,b}, but there is no GLB.

This is because c and d are incomparable.

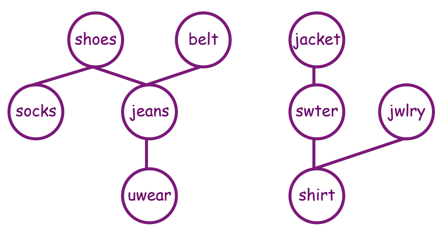
Similarly, a and b are upper bounds for {c,d}, but there is no LUB.



Total Orders

Consider the problem of getting dressed.

Precedence constraints are modeled by a poset in which a ≤ b iff you must put on a before b.



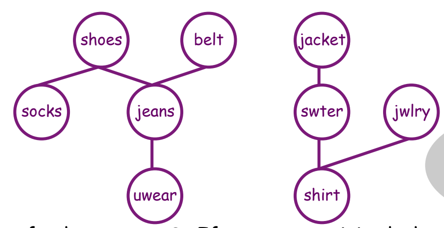
In what order will you get dressed while respecting constraints?

Let (S, ≤) be a poset (S finite). We will extend ≤ to a total order on S, so we can decide for all incomparable pairs whether to make a ≤ b, or vice versa w/o violating T,R,A.

Total Orders

Things we need:

Lemma: Every finite non-empty poset (S, ≤) has at least one minimal element.



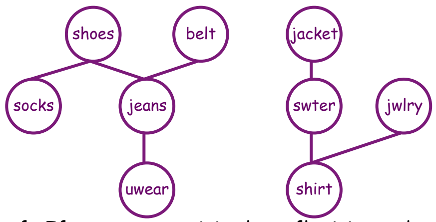
NOT A REAL PROOF! Should use induction!!

Proof: choose $a_0 \in S$. If a_0 was not minimal, then there exists $a_1 \leq a_0$, and so on until a minimal element is found.

Total Orders

More things we need:

Lemma: If (S, ≤) is a poset with a minimal, then (S-{a}, ≤) is also a poset.



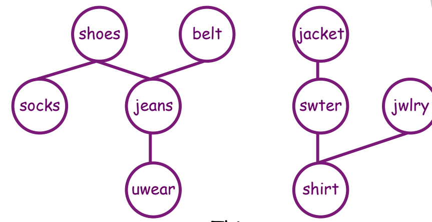
Proof: If you remove minimal a reflexivity and antisymmetry still hold. If x,y,z ∈ S-{a}, with x ≤ y and y ≤ z, then x ≤ z too, since (S, ≤) was transitive.

Total Orders

Think about what this means:

1. There is always a minimal element.
2. If you remove it you still have a poset.

Depending on which min elt is chosen each time, a different total order is obtained, but all TOs will be consistent with the PO.



This suggests:

alg Topological Sort
Input: poset (S, ≤)
Out: elements of S in total order
While S ≠ ∅
 Remove any min elt from S and output it.

Total Orders

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