University of Illinois at Chicago
Department of Computer Science

Final Examination

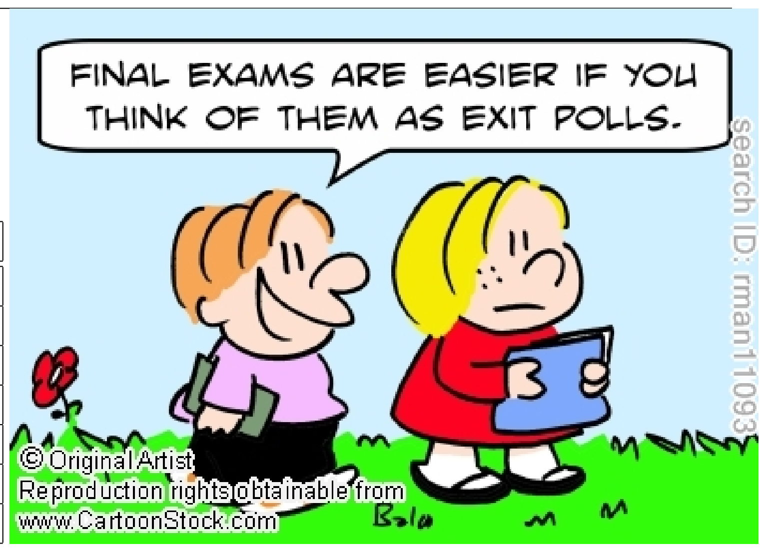
CS 151 Data Structures and Discrete Mathematics II
Fall 2012

3:30pm–5:30pm, Wednesday, December 12, 2012

Name:
Email:

- **Print your name and email, neatly in the space provided above; print your name at the upper right corner of *every page*. Please print legibly.**
- **This is a *closed book* exam. You are permitted to use only your brain. *Nothing else is permitted.***
- ***Show your work!* You will not get partial credit if we cannot figure out how you arrived at your answer.**
- **Write your answers in the space provided for the corresponding problem. Let us know if you need more paper.**
- **If any question is unclear, ask us for clarification.**
- **GOOD LUCK!**

Q	Pts	Score	Grader
1	29		
2	25		
3	10		
4	10		
5	25		
Total	99+1(free)		
Extra	2		



1. **Functions and sets.**

Circle correct answers. Each subitem is 1 point.

(a) Which of the following functions are bijections?

i. $f : \mathbb{R}^+ \to \mathbb{R}^+, f(x) = \frac{1}{1+x^2}$
ii. $f : \mathbb{R}^+ \to \mathbb{R}^+, f(x) = \log_2(x+1)$
iii. $f : \mathbb{Z}^+ \to \mathbb{Z}^+, f(x) = \binom{x}{1}$
iv. $f : \mathbb{Z} \to \mathbb{Z}, f(x) = x \mod 10^8$

(b) Let $A = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. Which of the following statements are true?

i. $\{\emptyset, \{\{\emptyset\}\}\}$ is the power set of A
ii. $A \cup \{\emptyset\} = \{\emptyset\}$
iii. $A \cup \{\emptyset\} = A$
iv. $A \cap \{\emptyset\} = \{\emptyset\}$
v. $A \cap \{\emptyset\} = A$
vi. Let $B = \{\emptyset, \{\emptyset\}\}$. Then $B \in A$ and $B \subseteq A$.

(c) Which of the following statements are **NOT** true?

i. If a function f is a *bijection* from set A to set B then there must be an inverse function f^{-1} such that $\forall a \in A, b \in B \quad f^{-1}(b) = a$ when $f(a) = b$.
ii. If a function is a *surjection* then its codomain must equal to its image.
iii. Let $f : S \to T$ and $g : T \to U$. If the composition $g \circ f$ is a bijection then f must be a bijection.
iv. A function must be an injection, a surjection, or both.
v. All relations are functions

(d) *(5 points)* Prove that if $S \subseteq T$ then $S \cup T = T$

(e) *(3 points each, 9 total)* 100 students are sick. 1000 students are tired. 10000 students are antsy for the break. Determine the number of students who are sick or tired or antsy for the break under each of the following conditions:

i. Every sick student is tired, and every tired student is antsy for the break.

ii. The sets of students are pairwise disjoint

iii. There are two students who are both sick and tired, two who are both tired and antsy of the break, two who are both sick and antsy for the break, and one who is sick and tired and antsy for the break.

2. **Probability and Counting.**
(5 points each, 25 total) The final exam of a discrete mathematics course consist of 50 true/false questions, each worth one point, and 25 multiple choice questions, each worth two points. (You can write your answer in terms of $C(n, k)$, $P(n, k)$, and factorials, for the appropriate values of n and k, without computing the actual numerical value of the answer.)
Briefly justify your answers.

(a) In how many ways can a student get a grade of 96 on the test? *Hint: this is not about probability, just counting.*

(b) Suppose the questions are numbered $1 \ldots 75$. In how many ways can a student get 2 even-numbered questions wrong?

(c) Suppose a lazy professor chooses 10 questions to be graded at random. In how many ways can the professor choose the questions to grade?

(d) Suppose a student has answered three questions wrong. Suppose a lazy professor chooses 10 questions to be graded at random. What is the probability that *at most* one of the chosen questions is wrong?

(e) Suppose the true/false questions are numbered $1 \ldots 50$ and the multiple choice questions are numbered $51 \ldots 75$. Suppose a student answers a true/false question with probability 0.9 and a multiple choice question with probability 0.8. What is the probability that the students gets a grade of 96 on the test? *Hint: see the first item*

3. **Binomial Theorem.**
(10 points) Prove the identity

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

using the Binomial Theorem: $\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$. (*Hint*: Recall, that for any x, $1^x = 1$).

4. **Combinatorial Identities.**
(10 points) Prove, using a combinatorial argument, that if n and k are integers with $1 \leq k \leq n$, then

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

(*Hint*: Show that the two sides of the identity count the number of ways to select a subset with k elements for a set with n elements and then an element of this subset).

5. **Relations.**
(3 points each, 12 total) Let R be a relation defined over binary strings in the following way:

- $(0,0) \in R$ and $(1,1) \in R$
- if $(a,b) \in R$ and $(c,d) \in R$ then $(ac, bd) \in R$, where ac is the concatenation of the strings a and c and bd is the concatenation of the strings b and d.
- nothing else is in R

(a) Which of the following pairs are in R (circle those that are)?
$(01, 01)$ $(01, 10)$ $(0001, 001)$ $(01101, 01101)$

(b) Is R reflexive? Justify your answer. If not, write the reflexive closure of R.

(c) Is R symmetric? Justify your answer. If not, write the symmetric closure of R.

(d) Is R anti-symmetric? Justify your answer.

(e) Is R transitive? Justify your answer.

EXTRA CREDIT Is R an equivalence relation? Justify your answer. If not, is one of the closures an equivalence relation? If so, which one.

EXTRA CREDIT Is R a partial order set? Justify your answer. If yes, draw a Hasse diagram of R.

(13 points) Still using the R from above (but just think of it a set of pairs), use induction to prove that for every $(x, y) \in R$ the length of x equals the length of y.

Base Case: *(State the base case)*

Inductive Hypothesis: Assume that for every $k < n$ for all strings s, t of length at most k, if $(s, t) \in R$ then the length of s equals the length of t

Inductive Step: Suppose $(x, y) \in R$ and the length of x and y is at most n. *(Continue the proof)*

Have a great break!