University of Illinois at Chicago
Department of Computer Science

Final Solution

CS 151 Data Structures and Discrete Mathematics II
Fall 2012

3:30pm–5:30pm, Wednesday, December 12, 2012

1. **Functions and sets.**

Circle correct answers. Each subitem is 1 point.

(a) Which of the following functions are bijections?

i. $f : \mathbb{R}^+ \to \mathbb{R}^+, f(x) = \frac{1}{1+x^2}$ **Solution: no**

ii. $f : \mathbb{R}^+ \to \mathbb{R}^+, f(x) = \log_2(x+1)$ **Solution: yes**

iii. $f : \mathbb{Z}^+ \to \mathbb{Z}^+, f(x) = \binom{x}{1}$ **Solution: yes**

iv. $f : \mathbb{Z} \to \mathbb{Z}, f(x) = x \mod 10^8$ **Solution: no**

(b) Let $A = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$. Which of the following statements are true?

i. $\{\emptyset, \{\{\emptyset\}\}\}$ is the power set of A **Solution: no**

ii. $A \cup \{\emptyset\} = \{\emptyset\}$ **Solution: no**

iii. $A \cup \{\emptyset\} = A$ **Solution: yes**

iv. $A \cap \{\emptyset\} = \{\emptyset\}$ **Solution: yes**

v. $A \cap \{\emptyset\} = A$ **Solution: no**

vi. Let $B = \{\emptyset, \{\emptyset\}\}$. Then $B \in A$ and $B \subseteq A$. **Solution: yes**

(c) Which of the following statements are **NOT** true?

i. If a function f is a *bijection* from set A to set B then there must be an inverse function f^{-1} such that $\forall a \in A, b \in B \quad f^{-1}(b) = a$ when $f(a) = b$. **Solution: true**

ii. If a function is a *surjection* then its codomain must equal to its image. **Solution: true**

iii. Let $f : S \to T$ and $g : T \to U$. If the composition $g \circ f$ is a bijection then f must be a bijection. **Solution: true**

iv. A function must be an injection, a surjection, or both. **Solution: no**

v. All relations are functions **Solution: no**

(d) *(5 points)* Prove that if $S \subseteq T$ then $S \cup T = T$

Solution: By definition of union, $T \subseteq S \cup T$. For each $x \in S \cup T$, either $x \in S$

or $x \in T$ (by definition of union). Since $S \subseteq T$ then for each $x \in S$ also $x \in T$ so $S \cup T \subseteq T$. Therefore, $S \cup T = T$.

(e) *(3 points each, 9 total)* 100 students are sick. 1000 students are tired. 10000 students are antsy for the break. Determine the number of students who are sick or tired or antsy for the break under each of the following conditions:

i. Every sick student is tired, and every tired student is antsy for the break.
Solution: Sick $\subseteq$ Tired $\subseteq$ Antsy. So 10000 total.

ii. The sets of students are pairwise disjoint
Solution: Pairwise disjoint sets so can use the sum rule. $100 + 1000 + 10000 = 11100$ total.

iii. There are two students who are both sick and tired, two who are both tired and antsy of the break, two who are both sick and antsy for the break, and one who is sick and tired and antsy for the break.
Solution: So there is one student who is sick and tired but not antsy, one who is tired and antsy but not sick, and one who is sick and antsy but not tired. And one who is all three. There are 4 students in the intersection. There are 97 students who are only sick, 997 who are only tired, and 9997 who are only antsy. Therefore the total is $97 + 997 + 9997 + 4 = 11095$

2. **Probability and Counting.**

(5 points each, 25 total) The final exam of a discrete mathematics course consist of 50 true/false questions, each worth one point, and 25 multiple choice questions, each worth two points. (You can write your answer in terms of $C(n,k)$, $P(n,k)$, and factorials, for the appropriate values of n and k, without computing the actual numerical value of the answer.)
Briefly justify your answers.

(a) In how many ways can a student get a grade of 96 on the test? *Hint: this is not about probability, just counting.*
Solution: $C(50,4) + C(50,2)C(25,1) + C(25,2)$

(b) Suppose the questions are numbered $1 \ldots 75$. In how many ways can a student get 2 even-numbered questions wrong?
Solution: $\binom{37}{2}$

(c) Suppose a lazy professor chooses 10 questions to be graded at random. In how many ways can the professor choose the questions to grade?
Solution: $\binom{75}{10}$

(d) Suppose a student has answered three questions wrong. Suppose a lazy professor chooses 10 questions to be graded at random. What is the probability that *at most* one of the chosen questions is wrong?
Solution: $\frac{C(72,10)+3C(72,9)}{C(75,10)}$

(e) Suppose the true/false questions are numbered $1 \ldots 50$ and the multiple choice questions are numbered $51 \ldots 75$. Suppose a student answers a true/false question with probability 0.9 and a multiple choice question with probability 0.8. What is the probability that the students gets a grade of 96 on the test? *Hint: see the first item*

Solution:
$C(50,4)0.9^{46}0.1^{4}0.8^{25} + C(50,2)C(25,1)0.9^{48}0.1^{2}0.8^{24}0.2 + C(25,2)0.9^{50}0.8^{23}0.2^{2}$

3. **Binomial Theorem.**
(10 points) Prove the identity

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

using the Binomial Theorem: $\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$. (*Hint*: Recall, that for any x, $1^x = 1$).
Solution: $a = -1, b = 1$.
So $(a+b)^n = (-1+1)^n = 0$.
$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$.

4. **Combinatorial Identities.**
(10 points) Prove, using a combinatorial argument, that if n and k are integers with $1 \leq k \leq n$, then

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

(*Hint*: Show that the two sides of the identity count the number of ways to select a subset with k elements for a set with n elements and then an element of this subset).
Solution: Suppose we are choosing a team of k people out of n total and a team captain. We can first choose the team in $\binom{n}{k}$ ways. For every choice of the team, we can then choose one of the k members to be the captain in $\binom{k}{1} = k$ ways. Using product rule, we get the LHS.
We can also first choose the captain out of all the n people in n ways and then choose the remaining $k-1$ team members out of the remaining $n-1$ people. Using the product rule we get RHS.

5. **Relations.**
(3 points each, 12 total) Let R be a relation defined over binary strings in the following way:

- $(0,0) \in R$ and $(1,1) \in R$
- if $(a,b) \in R$ and $(c,d) \in R$ then $(ac, bd) \in R$, where ac is the concatenation of the strings a and c and bd is the concatenation of the strings b and d.
- nothing else is in R

(a) Which of the following pairs are in R (circle those that are)?
$(01, 01)$ $(01, 10)$ $(0001, 001)$ $(01101, 01101)$ Solution: $(01, 01)$ and $(01101, 01101)$

(b) Is R reflexive? Justify your answer. If not, write the reflexive closure of R.
Solution: Yes, (by induction): $(0, 0)$ and $(1, 1)$ are in R and for every (ac, bd) if $a = b$ and $c = d$ (by IH) then $ac = bd$. So for all elements $(x, y) \in R$ it is true that $x = y$.

(c) Is R symmetric? Justify your answer. If not, write the symmetric closure of R.
Solution: Yes, since the only elements in R are of the type (x, x), so only self-loops

(d) Is R anti-symmetric? Justify your answer.
Solution: Yes, since only (x, x) are in R so (vacuously) for every $(x, y) \in R$ if $(y, x) \in R$ then $x = y$.

(e) Is R transitive? Justify your answer.
Solution: Yes, since only self loops are in R so no paths of length 2.

EXTRA CREDIT Is R an equivalence relation? Justify your answer. If not, is one of the closures an equivalence relation? If so, which one.
Solution: Yes, since refexive, symmetric, transitive.

EXTRA CREDIT Is R a partial order set? Justify your answer. If yes, draw a Hasse diagram of R.
Solution: Yes, since reflexive, antisymmetric, transitive.

(13 points) Still using the R from above (but just think of it a set of pairs), use induction to prove that for every $(x, y) \in R$ the length of x equals the length of y.

Base Case: *(State the base case)*
Holds for $(0, 0)$ since $length(0) = length(0) = 1$. Holds for $(1, 1)$ since $length(1) = length(1) = 1$.

Inductive Hypothesis: Assume that for every $k < n$ for all strings s, t of length at most k, if $(s, t) \in R$ then the length of s equals the length of t

Inductive Step: Suppose $(x, y) \in R$ and the length of x and y is at most n. *(Continue the proof)*
Since $(x, y) \in R$ then there must exist $(a, b) \in R$ and $(c, d) \in R$ such that $x = ac$ and $y = bd$. Since the length of a, b, c, d is less than n, by Inductive Hypothesis, $length(a) = length(b)$ and $length(c) = length(d)$. So $length(x) = length(ac) = length(a) + length(c) = length(b) + length(d) = length(bd) = length(y)$.

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Have a great break!