CS202 Fall 2012
Lecture 13 – 10/11

Solving Recurrences

Each wife of Fibonacci,
Eating nothing that wasn't starchy,
Weighed as much as two before her.
His fifth was some signora!

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Recurrences

We've seen a variety. Techniques for solving depend on the particular form of the recurrence.

Arithmetic sequence:

$a_n = a_{n-1} + d$
$= a_{n-2} + d + d$
$= a_{n-3} + d + d + d$
...
$= a_{n-n} + d + d + d + \ldots + d$
$= a_0 + nd$

Recurrences

Geometric sequence:

$a_n = ra_{n-1}$
$= rra_{n-2}$
$= rrra_{n-3}$
...
$= rr\ldots ra_{n-n}$
$= r^n a_0$

Recurrences

We've seen a variety. Techniques for solving depend on the particular form of the recurrence.

Linear Recurrences with Constant Coefficients

$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + f(n)$

"order k"

Ex. $a_n = 5a_{n-1} - 6a_{n-2}$, $a_0 = 0$, $a_1 = 1$

$a_2 = 5$
$a_3 = 5(5) - 6(1) = 19$
$a_4 = \ldots$

Given any k consecutive values, a unique solution exists.

Characteristic Equation

Lemma: Let $c_1, c_2, \ldots, c_k$ be numbers. A recurrence relation of the form

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$$

is satisfied by the sequence

$$1, r, r^2, r^3, \ldots, r^n, \ldots$$

where r is a non-zero real number, if and only if r satisfies the equation

$$r^k - c_1r^{k-1} - c_2r^{k-2} - \ldots - c_k = 0$$

Characteristic equation of

5

Recurrences

Linear homogeneous recurrence relations with constant coefficients.

another example:

$T(n) = 7T(n-1) - 10T(n-2)$

$T(0) = 2,\ T(1) = 1$

We want to be able to compute T(n), without having to compute T(n-k)

To achieve this we "solve" the recurrence.

- Rewrite: $T(n) - 7T(n-1) + 10T(n-2) = 0$
- Find characteristic eqn: $r^2 - 7r + 10 = 0$
- Find roots of char eqn: $(r-2)(r-5) = 0$, $r=2,5$
- General solution is $T(n) = A2^n + B5^n$

Recurrences

Example continued:

$T(n) = 7T(n-1) - 10T(n-2)$

$T(0) = 2,\ T(1) = 1$

- General solution is $T(n) = A2^n + B5^n$

Now what? … think about what we know!

$A2^0 + B5^0 = 2$	$A + B = 2$	$A = 3,\ B = -1$
$A2^1 + B5^1 = 1$	$2A + 5B = 1$	

- Unique solution is $T(n) = 3 \cdot 2^n - 5^n$

Recurrences

You try a tricky one:

$T(n) = T(n-2)/4$

$T(0) = 1,\ T(1) = 0$

- Rewrite: $T(n) - \frac{1}{4}T(n-2) = 0$
- Find characteristic eqn: $r^2 - \frac{1}{4} = 0$
- Find roots of char eqn: $r = \frac{1}{2}, -\frac{1}{2}$
- General solution is: $T(n) = A(\frac{1}{2})^n + B(-\frac{1}{2})^n$
- Set up system of eqn to get unique soln. $T(0) = 1 = A + B$, $T(1) = 0 = A^* \frac{1}{2} - B^* \frac{1}{2}$, $A = \frac{1}{2},\ B = \frac{1}{2}$

$$T(n) = (\tfrac{1}{2})^{n+1} + (-1)^n (\tfrac{1}{2})^{n+1}$$

$$T(n) = \begin{cases} \frac{1}{2^n} & n\ even \\ 0 & n\ odd \end{cases}$$

Recurrences

Suppose the characteristic eqn factors into:

$$(r-2)^3(r-3)^2(r-5) = 0$$

Roots are 2,2,2,3,3,5

If you have a non-distinct root r of multiplicity m, then $n^i r^n$, i=0,1,...m-1 are all solutions.

In this case the general solution is:

$$T(n) = A_1n^2 2^n + A_2 n 2^n + A_3 2^n + A_4 n 3^n + A_5 3^n + A_6 5^n$$

$$T(n) = (A_1 n^2 + A_2 n + A_3)2^n + (A_4 n + A_5)3^n + A_6 5^n$$

To find the unique particular solution, we would need the 6 boundary values. Solve for $A_1, \ldots A_6$ by solving 6 equations in 6 unknowns.

Recurrences

Claim:
If for all $n \geq 3$, $a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3}$, and $a_0 = 0, a_1 = 1, a_2 = 2$ then for all $n \geq 0$, $a_n = n$

Proof: By induction on n.

Base Cases: For n=0, 1, 2 we check that $a_n = n$. Indeed it holds by

Inductive Step:
$a_n = 4a_{n-1} - 5a_{n-2} + 2a_{n-3}$ by definition of the recurrence
$= 4(n-1) - 5(n-2) + 2(n-3)$ by Inductive Hypothesis on $a_{n-1}, a_{n-2}, a_{n-3}$
$= 4n - 4 - 5n + 10 + 2n - 6$ by algebra
$= n$ by algebra

Conclusion: Since the Base Cases and the Inductive Step hold, the Claim is true for all $n \geq 0$,

Recurrences

Linear NONhomogeneous recurrence relations with constant coefficients.

$$c_0a_n + c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} = f(n),$$

Where f(n) is
- constant
- polynomial in n
- c^n for some constant c
- $c^n \cdot$ polynomial(n)

This is not in the book. Notes are posted.

Recurrences

First, some notation:

A sequence $a_0, a_1, a_2, \ldots,$ is denoted $\langle a_n \rangle$

Examples
- $\langle 2^n \rangle = 1,2,4,8,\ldots$
- $\langle n^2 \rangle = 0, 1,4,9,\ldots$
- $\langle n \rangle = 0, 1,2,3,\ldots$

Note: if $\langle a_n \rangle$ and $\langle b_n \rangle$ are sequences, then $\langle a_n \rangle + \langle b_n \rangle$ represents the sequence $\langle a_n + b_n \rangle$ (termwise addition).

Recurrences

Sequence operators:

- Constant multiplication
  $c \cdot \langle a_n \rangle$ defined to be $\langle c \cdot a_n \rangle$
  Ex: $3 \cdot \langle 2^n \rangle = \langle 3 \cdot 2^n \rangle = 3, 6, 12, 24, 48, \ldots$
- Shift "E"
  $E\langle a_n \rangle = \langle a_{n+1} \rangle$ shifts sequence to left
  Ex: $E\langle 2^n \rangle = \langle 2^{n+1} \rangle = 2, 4, 8, 16, \ldots$
  Ex: $E\langle 3n + 1 \rangle = \langle 3(n+1) + 1 \rangle = \langle 3n + 4 \rangle$

	a_0	a_1	a_2	a_3
$\langle a_n \rangle$	1	2	4	8
$E\langle a_n \rangle$	2	4	8	16

Recurrences

Combining operators:

- If A,B are seq ops, then A+B is a seq op:
  $(A+B)\langle a_n \rangle$ defined to be $A\langle a_n \rangle + B\langle a_n \rangle$
  Ex: $(E+2)\langle 2^n \rangle = E\langle 2^n \rangle + 2\langle 2^n \rangle$
  $= \langle 2^{n+1} \rangle + \langle 2 \cdot 2^n \rangle$
  $= \langle 2^{n+1} \rangle + \langle 2^{n+1} \rangle$
  $= \langle 2^{n+1} + 2^{n+1} \rangle$
  $= \langle 2 \cdot 2^{n+1} \rangle = \langle 2^{n+2} \rangle$
- If A,B are seq ops, then AB is a seq op:
  $(AB)\langle a_n \rangle$ defined to be $A(B\langle a_n \rangle)$
  Ex: $E^3\langle a_n \rangle = E \cdot E \cdot E\langle a_n \rangle = E(E(E\langle a_n \rangle)) = \langle a_{n+3} \rangle$

Recurrences

An important operator: (E-1)

- Example: $(E-1)\langle n \rangle = E\langle n \rangle - \langle n \rangle$
  $= \langle n + 1 \rangle - \langle n \rangle$
  $= \langle n + 1 - n \rangle = \langle 1 \rangle$
- More generally: $(E-1)\langle a_n \rangle = \langle a_{n+1} \rangle - \langle a_n \rangle$
  $= \langle a_{n+1} - a_n \rangle$

Discrete derivative

- $(E-1)\langle n^2 \rangle = E\langle n^2 \rangle - \langle n^2 \rangle$
  $= \langle (n+1)^2 \rangle - \langle n^2 \rangle$
  $= \langle n^2 + 2n + 1 \rangle - \langle n^2 \rangle = \langle 2n + 1 \rangle$

Recurrences

For any constant sequence $\langle c \rangle$:

$(E-1)\langle c \rangle = \langle 0 \rangle$

(E-1) "annihilates" $\langle c \rangle$

- TRICK: to solve a **NONhomogeneous** linear recurrence with constant coefficients, turn it into a **homogeneous** recurrence by applying operators to annihilate the right side.

Recurrences

Example: solve $a_n = 5a_{n-1} - 6a_{n-2} + 4$

- Rewrite: $a_n - 5a_{n-1} + 6a_{n-2} = 4$
- Rewrite again so n is smallest index:
  $a_{n+2} - 5a_{n+1} + 6a_n = 4$
- Rewrite again as a sequence:
  $\langle a_{n+2} - 5a_{n+1} + 6a_n \rangle = \langle 4 \rangle$
- Rewrite again using operators:
  $(E^2 - 5E + 6)\langle a_n \rangle = \langle 4 \rangle$

(E-1) "annihilates" $\langle c \rangle$

Recurrences

Example: solve $a_n = 5a_{n-1} - 6a_{n-2} + 4$

$(E^2 - 5E + 6)\langle a_n \rangle = \langle 4 \rangle$

(E-1) "annihilates" $\langle c \rangle$

- Annihilate right side:
  $(E-1)(E^2 - 5E + 6)\langle a_n \rangle = (E-1)\langle 4 \rangle$
  $(E^3 - 6E^2 + 11E - 6)\langle a_n \rangle = \langle 0 \rangle$
- But that's just:
  $a_{n+3} - 6a_{n+2} + 11a_{n+1} - 6a_n = 0$
  $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$

Homogeneous!!!

Recurrences

Example: solve $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$

Homogeneous!!!

- Characteristic equation:
  $(r^3 - 6r^2 + 11r - 6) = 0$
  $(r-1)(r-2)(r-3) = 0$
- General solution: $a_n = A_1 + A_2 2^n + A_3 3^n$
- Not a coincidence that characteristic equation looks the same as the annihilating operator:
  $f(r)=0$ corresponds exactly to $g(f)\langle a_n \rangle = \langle 0 \rangle$

Recurrences

Technique:

- Rewrite recurrence in sequence notation.
- Rewrite left side as $OP\langle a_n \rangle$.
- Find an annihilator for sequence on right side and apply to both sides
- Read characteristic equation off left side.
- Solve homogeneous equation as before.

How?

Recurrences

So far we know how to annihilate ⟨c⟩ for constant c.

Linear NONhomogeneous recurrence relations with constant coefficients.

$$c_0a_n + c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k} = f(n),$$

Where f(n) is
- constant
- polynomial in n
- c^n for some constant c
- $c^n \cdot$ polynomial(n)

Use operator (E - 1)
"Discrete derivative"

Recurrences

Group challenge: find the annihilators for the remaining function types.

$$(E - 1)\langle a_n\rangle = E\langle a_n\rangle - \langle a_n\rangle = \langle a_{n+1}\rangle - \langle a_n\rangle = \langle a_{n+1} - a_n\rangle$$

Linear NONhomogeneous recurrence relations with constant coefficients.

$$c_0a_n + c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k} = f(n),$$

Where f(n) is
- constant
- polynomial in n
- c^n for some constant c
- $c^n \cdot$ polynomial(n)

Use operator (E - 1)
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Recurrences

Group challenge: find the annihilators for the remaining function types.

$$(E - 1)\langle a_n\rangle = E\langle a_n\rangle - \langle a_n\rangle = \langle a_{n+1}\rangle - \langle a_n\rangle = \langle a_{n+1} - a_n\rangle$$

Linear NONhomogeneous recurrence relations with constant coefficients.

$$c_0a_n + c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k} = f(n),$$

Where f(n) is
- constant
- polynomial in n of degree k

Use operator (E - 1)
"Discrete derivative"

Annihilator: $(E - 1)^{k+1}$

Recurrences

How do you annihilate $\langle c^n\rangle$?

Consider $\langle 3^n\rangle$:

$E\langle 3^n\rangle =$ 3, 9, 27, …

$3\langle 3^n\rangle =$ 3, 9, 27, …

So $(E-3)\langle 3^n\rangle = \langle 0\rangle$

In general (E-c) annihilates $\langle c^n\rangle$

Recurrences

Table of handy annihilators:

Sequence	Annihilator
$\langle c \rangle$	E–1
$\langle$polynomial(n) of degree $\le$ k$\rangle$	$(E–1)^{k+1}$
$\langle c^n \rangle$	E–c
$\langle c^n \cdot$polynomial(n) of degree $\le$ k$\rangle$	$(E–c)^{k+1}$

Recurrences

Another little helpful fact:

Suppose operator A annihilates $\langle a_n \rangle$
And B annihilates $\langle b_n \rangle$

Then AB annihilates $\langle a_n + b_n \rangle$

Recurrences

Solve $a_n = 2a_{n-1} + 2^n - 1$, $a_0 = 0$.

Rewrite as sequences: $\langle a_n - 2a_{n-1} \rangle = \langle 2^n - 1 \rangle$

$\langle a_{n+1} - 2a_n \rangle = \langle 2^{n+1} - 1 \rangle$

Rewrite using operators: $(E-2)\langle a_n \rangle = \langle 2^{n+1} - 1 \rangle$

Rewrite right side as sum: $(E-2)\langle a_n \rangle = \langle 2^{n+1} \rangle - \langle 1 \rangle$

Apply annihilators to both sides:

$(E-2)(E-1)(E-2)\langle a_n \rangle = (E-2)(E-1)(\langle 2^{n+1} \rangle - \langle 1 \rangle) = \langle 0 \rangle$

Write down general soln: $a_n = (A_1 + A_2 n)2^n + A_3$

Recurrences

Solve $a_n = 2a_{n-1} + 2^n - 1$, $a_0 = 0$.

General solution: $a_n = (A_1 + A_2 n)2^n + A_3$

Need 3 initial values because we have 3 unknowns to solve for.

$a_1 = 2(0) + 2^1 - 1 = 1$, $a_2 = 2(1) + 2^2 - 1 = 5$

Solve 3 equations in 3 unknowns (using the general solution):

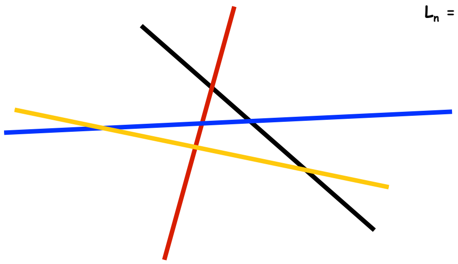
$A_1 + A_3 = 0$
$2A_1 + 2A_2 + A_3 = 1$
$4A_1 + 8A_2 + A_3 = 5$

$A_1 = -1$, $A_2 = A_3 = 1$

$a_n = (n - 1)2^n + 1$

Example

The max number of regions n lines divide the plane (L_n):



$$L_n = L_{n-1} + 1 + (n-1) = L_{n-1} + n$$

$$\langle L_n - L_{n-1} \rangle = \langle n \rangle$$

$$\langle L_{n+1} - L_n \rangle = \langle n+1 \rangle$$

$$(E-1)\langle L_n \rangle = \langle n+1 \rangle$$

$$(E-1)^2(E-1)\langle L_n \rangle = \langle 0 \rangle$$

$$(E-1)^3 \langle L_n \rangle = \langle 0 \rangle$$

$$L_n = A_1 + A_2 n + A_3 n^2$$

$1 = A_1$

$2 = A_1 + A_2 + A_3$ $\Rightarrow A_1 = 1$, $A_2 = \frac{1}{2}$, $A_3 = \frac{1}{2}$ $\Rightarrow L_n = 1 + n/2 + n^2/2$

$4 = A_1 + 2A_2 + 4A_3$

Fibonacci

$F_n = F_{n-1} + F_{n-2}$ $\quad F_0 = 1, F_1 = 1$

$\langle F_n - F_{n-1} - F_{n-2} \rangle = \langle 0 \rangle$

$\langle F_{n+2} - F_{n+1} - F_n \rangle = \langle 0 \rangle$

$\langle F_{n+2} - F_{n+1} - F_n \rangle = \langle 0 \rangle$

$(E^2 - E - 1)\langle F_n \rangle = \langle 0 \rangle$

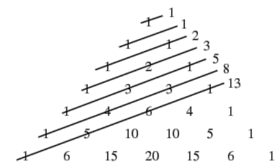
$(E - \varphi)(E - \varphi')\langle F_n \rangle = \langle 0 \rangle$ **where** $\varphi = \frac{1+\sqrt{5}}{2}$ *and* $\varphi' = \frac{1-\sqrt{5}}{2}$

$$F_n = A_1\varphi^n + A_2\varphi'^n$$

$$1 = A_1 + A_2$$

$$1 = A_1\frac{1+\sqrt{5}}{2} + A_2\frac{1-\sqrt{5}}{2}$$

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$



Recurrences

Solve $a_n = 2a_{n/2} + n-1$, $a_1 = 0$.

Mergesort
# of comparisons