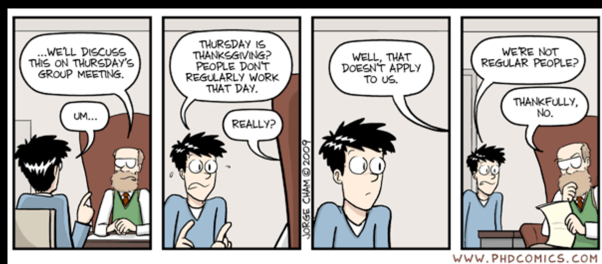
CS202 Fall 2011
Lecture (lost count) – 12/1

Relations

Prof. Tanya Berger-Wolf



Relations

Recall the definition of the Cartesian (Cross) Product:

The Cartesian Product of sets A and B, A x B, is the set

A x B = {(a,b) : a∈A and b∈B}.

aRb or **(a,b) ∈ R** means **"*a* is related to *b*"**

A relation is just any subset of the Cartesian Product:

R ⊆ AxB

- Ex1: A = {0,1,2}, B = {2,3} => AxB = {(0,2), (0,3), (1,2), (1,3), (2,2), (2,3)}
  R = {(a,b) | a < b}. So R = {(0,2), (0,3), (1,2), (1,3), (2,3)} = AxB-{(2,2)}
- Ex2: A = students at UIC; B = courses at UIC.
  R = {(a,b) | student a is enrolled in class b}
- Ex3: A = {3 letter strings}, B = {all English words}
  R = {(a,b) | a is a prefix of b}

Relations and Functions

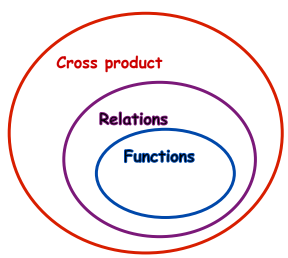
Recall the definition of a function:

f = {(a,b) : b = f(a) , a∈A and b∈B}

Is every function a relation?

Yes, a function is a special kind of relation.

Draw Venn diagram of **cross products, relations, functions**



Properties of Relations

Reflexivity:

A relation R on AxA is reflexive if for all a∈A, (a,a) ∈R.

Symmetry:

A relation R on AxA is symmetric if (a,b) ∈ R implies (b,a) ∈ R .

Properties of Relations

Transitivity:

A relation on AxA is transitive if
(a,b) ∈ R and (b,c) ∈ R imply (a,c) ∈ R.

Anti-symmetry [Epp p.632]:

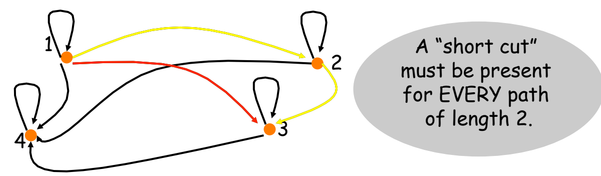
A relation on AxA is anti-symmetric if
(a,b) ∈ R implies (b,a) ∉ R.

Properties of Relations - techniques…

How can we check for **transitivity**?
Draw a picture of the relation (called a "graph") [Epp p. 580].

- **Vertex** for every element of A
- **Edge** for every element of R

R={(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}

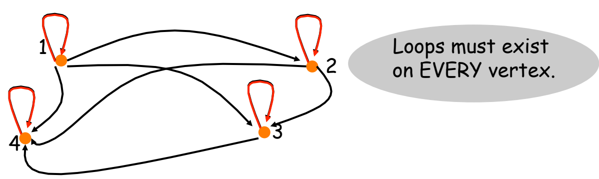


Properties of Relations - techniques…

How can we check for the **reflexive property**?
Draw a picture of the relation (called a "graph").

- Vertex for every element of A
- Edge for every element of R

R={(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}

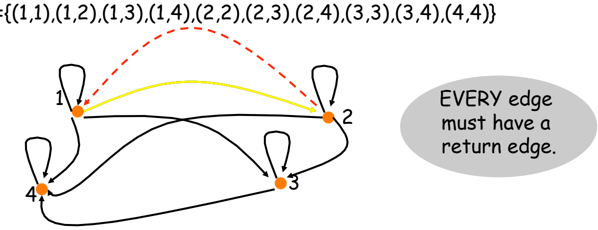


Properties of Relations - techniques…

How can we check for the symmetric property?
Draw a picture of the relation (called a "graph").

- Vertex for every element of A
- Edge for every element of R

R={(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}

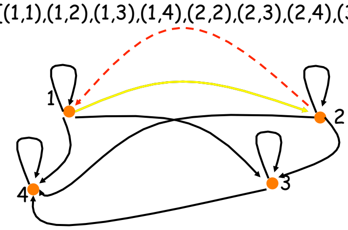


Properties of Relations - techniques…

How can we check for the **anti-symmetric property**?
Draw a picture of the relation (called a "graph").

- Vertex for every element of A
- Edge for every element of R

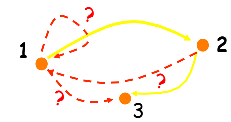
R={(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}



NO edge can have a return edge.

Properties of Relations - techniques…

Let R be a relation on People,
R={(x,y): x and y have lived in the same country}



Is R transitive? No

Is it symmetric? Yes

Is it reflexive? Yes

Is it anti-symmetric? No

Properties of Relations - techniques…

Let R be a relation on positive integers,
R={(x,y): 3|(x-y)}

Suppose (x,y) and (y,z) are in R.

Then we can write 3j = (x-y) and 3k = (y-z)

Definition of "divides"

Can we say 3m = (x-z)? Is (x,z) in R?

Add prev eqn to get: 3j + 3k = (x-y) + (y-z)

3(j + k) = (x-z)

Is R transitive? Yes

Properties of Relations - techniques…

Let R be a relation on positive integers,
R={(x,y): 3|(x-y)}

Is (x,x) in R, for all x?

Does 3k = (x-x) for some k?

Definition of "divides"

Yes, for k=0.

Is R transitive? Yes

Is it reflexive? Yes

Properties of Relations - techniques…

Let R be a relation on positive integers,
R={(x,y): 3|(x-y)}

Suppose (x,y) is in R.

Then 3j = (x-y) for some j.

Definition of "divides"

Does 3k = (y-x) for some k?

Yes, for k=-j.

Is R transitive? Yes

Is it symmetric? Yes

Is it reflexive? Yes

Properties of Relations - techniques…

Let R be a relation on positive integers,
R={(x,y): 3|(x-y)}

Suppose (x,y) is in R.

Then 3j = (x-y) for some j.

Definition of "divides"

Does 3k = (y-x) for some k?

Yes, for k=-j.

Is R transitive? Yes

Is it symmetric? Yes

Is it reflexive? Yes

Is it anti-symmetric? No

More than one relation

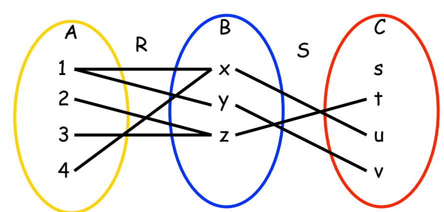
Suppose we have 2 relations, R_1 and R_2, and recall that relations are just sets! So we can take unions, intersections, complements, symmetric differences, etc.

There are other things we can do as well…

More than one relation

Let R be a relation from A to B ($R \subseteq A \times B$), and let S be a relation from B to C ($S \subseteq B \times C$). The composition of R and S is the relation from A to C ($S \circ R \subseteq A \times C$):

$S \circ R = \{(a,c): \exists b \in B, (a,b) \in R, (b,c) \in S\}$



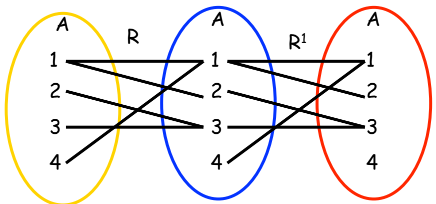
$S \circ R = \{(1,u),(1,v),(2,t),(3,t),(4,u)\}$

More than one relation

Let R be a relation on A. Inductively define

$R^1 = R$

$R^n = R^{n-1} \circ R$



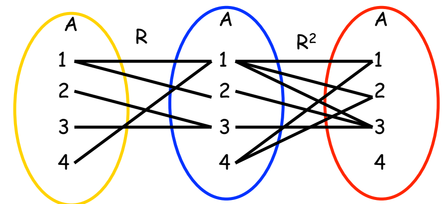
$R^2 = R^1 \circ R = \{(1,1),(1,2),(1,3),(2,3),(3,3),(4,1),(4,2)\}$

More than one relation

Let R be a relation on A. Inductively define

$R^1 = R$

$R^n = R^{n-1} \circ R$



$R^3 = R^2 \circ R = \{(1,1),(1,2),(1,3),(2,3),(3,3),(4,1),(4,2),(4,3)\}$

… = R4

= R5

= R6…

Relations - A Theorem:

If R is a transitive relation, then $R^n \subseteq R$, $\forall n$.

Proof by induction on n.

Base case (n=1): $R^1 \subseteq R$ because by definition, $R^1 = R$.

Inductive Step: IH: if R is transitive, then $R^{n-1} \subseteq R$.

Prove: if R is transitive, then $R^n \subseteq R$.

We are trying to prove that $R^n \subseteq R$. To do this, we select an element of R^n and show that it is also an element of R.

Let (a,b) be an element of R^n. Since $R^n = R^{n-1} \circ R$, we know there is an x so that $(a,x) \in R$ and $(x,b) \in R^{n-1}$.

By IH, since $R^{n-1} \subseteq R$, $(x,b) \in R$.

But wait, if $(a,x) \in R$, and $(x,b) \in R$, and R is transitive, then $(a,b) \in R$.

Typical way of proving subset.

Relations - A Theorem:

If R is a **transitive** relation, then $R^n \subseteq R$, $\forall n$.

Aside: notice that this theorem allows us to conclude that the previous relation was NOT transitive.

Recall: "if p then q" ≡ "if not q then not p."

We saw that R^n was not a subset of R (it was growing on every iteration).

Therefore, R is not transitive.

Relations - more techniques…

Suppose we have our old relation R on AxB, where
A={1,2,3,4}, and B={u,v,w},
R={(1,u),(1,v),(2,w),(3,w),(4,u)}.

Then we can represent R as:

	u	v	w
1	1	1	0
2	0	0	1
3	0	0	1
4	1	0	0

The labels on the outside are for clarity. It's really the matrix in the middle that's important.

This is a |A| x |B| matrix whose entries indicate membership in R.

Relations - more techniques…

Some things to think about.
Let R be a relation on a set A, and let M_R be the matrix representation of R.
Then R is reflexive if, _______________.

	u	v	w
u	1	1	0
v	0	1	1
w	0	0	1

A. All entries in M_R are 1.
B. The \ diagonal of M_R contains only 1s.
C. The first column of M_R contains no 0s.
D. None of the above.

Relations - more techniques…

Some things to think about.
Let R be a relation on a set A, and let M_R be the matrix representation of R.
Then R is symmetric if, _______________.

	u	v	w
u	1	0	1
v	0	0	1
w	1	1	0

A. All entries above the \ are 1.
B. The first and last columns of M_R contain an equal # of 0s.
C. M_R is visually symmetric about the \ diagonal.
D. None of the above.

Relations - more techniques…

Suppose we have R1 and R2 defined on A:

R_1	u	v	w
u	1	0	1
v	0	0	1
w	1	1	0

R_2	u	v	w
u	1	1	0
v	0	1	1
w	0	0	1

Then $R_1 \cup R_2$ is the bitwise "or" of the entries: $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$M_{R1 \cup R2} = M_{R1} \vee M_{R2}$

Then $R_1 \cap R_2$ is the bitwise "and" of the entries: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$M_{R1 \cap R2} = M_{R1} \wedge M_{R2}$

Closure

Consider relation R={(1,2),(2,2),(3,3)} on the set *A* = {1,2,3,4}.
Is R reflexive?

No

- What can we add to R to make it reflexive?

(1,1), (4,4)

R' = R U {(1,1),(4,4)} is called the *reflexive closure* of R.

Closure

Definition:
The closure of relation R on set *A* with respect to property P is the relation R' with

1. R ⊆ R'
2. R' has property P
3. ∀S with R ⊆ S and S has property P, R' ⊆ S.

P-Closure for a relation might not exist!

If relation R has property P then R' = R.

Reflexive Closure

Let r(R) denote the reflexive closure of relation R.

Then r(R) = R U { (a,a): ∀ a ∈ *A* }

Fine, but does that satisfy the definition?

1. R ⊆ r(R) We added edges!
2. r(R) is reflexive By defn
3. Need to show that for any S with particular properties, r(R) ⊆ S.

Let S be such that R ⊆ S and S is reflexive. Then

{(a,a): ∀ a ∈ *A* } ⊆ S (since S is reflexive) and R⊆S (given). So, r(R) ⊆ S.

Symmetric Closure

Let s(R) denote the symmetric closure of relation R.

Then s(R) = R U { (b,a): (a,b) ∈ R }

Fine, but does that satisfy the definition?

1. R ⊆ s(R) We added edges!
2. s(R) is symmetric By defn
3. Need to show that for any S with particular properties, s(R) ⊆ S.

Let S be such that R ⊆ S and S is symmetric. Then

{(b,a): (a,b) ∈ R } ⊆ S (since S is symmetric) and R⊆S (given). So, s(R) ⊆ S.

Transitive Closure (Epp p.587)

Let c(R) denote the transitive closure of relation R.

Then c(R) = R U { (a,c): ∃b (a,b),(b,c) ∈ R }

- Example: A={1,2,3,4}, R={(1,2),(2,3),(3,4)}.

Apply definition to get:

c(R) = {(1,2),(2,3),(3,4), (1,3), (2,4) }

Which of the following is true:

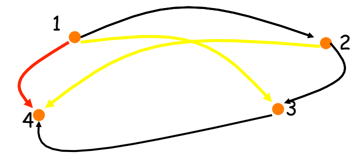
a) This set is transitive, but we added too much.
b) This set is the transitive closure of R.
c) This set is not transitive, one pair is missing.
d) This set is not transitive, more than 1 pair is missing.

Transitive Closure

So how DO we find the transitive closure?

Draw a graph.

- Example: A={1,2,3,4}, R={(1,2),(2,3),(3,4)}.



- Define a *path* in a relation R, on A to be a sequence of elements from A: $a, x_1, \ldots x_i, \ldots x_{n-1}, b$, with $(a, x_1) \in R$, $\forall i\ (x_i, x_{i+1}) \in R$, $(x_{n-1}, b) \in R$.

"Path from a to b."

Transitive Closure

Formally:

If t(R) is the transitive closure of R, and if R contains a path from a to b, then (a,b) ∈ t(R)

Note:

- Later classes (401) will give you efficient algorithms for determining if there is a path between two vertices in a graph (graph connectivity problem)

Equivalence Relations

Example:

Let S = {people in this classroom}, and let

R = {(a,b): a has same # of coins in his/her bag as b}

This is a special kind of relation, characterized by the properties it has.

What's special about it?

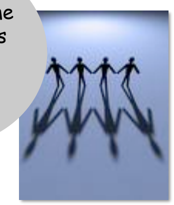
Quiz time:

Is R reflexive? Yes

Is R symmetric? Yes

Is R transitive? Yes

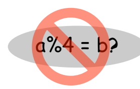
Everyone with the same # of coins as you is just like you.



Equivalence Relations

Formally:
Relation R on A is an ***equivalence relation*** if R is
Reflexive (∀ a ∈ A, aRa)
Symmetric (aRb --> bRa)
Transitive (aRb ^ bRc --> aRc)

Example:
S = Z (integers), R = {(a,b) : a ≡ b mod 4}
Is this relation an equivalence relation on S?
Have to PROVE reflexive, symmetric, transitive.



Equivalence Relations

Example:
S = Z (integers), R = {(a,b) : a ≡ b mod 4}
Is this relation an equivalence relation on S?

Start by thinking of R a different way: aRb iff there is an int k so that a = 4k + b. Your quest becomes one of finding ks.

Let a be any integer. aRa since a = 4·0 + a.
Consider aRb. Then a = 4k + b. But b = -4k + a.
Consider aRb and bRc. Then a = 4k + b and b = 4j + c. So, a = 4k + 4j +c = 4(k+j) + c.

Equivalence Relations

Example:
S = people in this room,
R = {(a,b) : total \$ on a is within \$1.00 of total \$ on b}
Is this relation an equivalence relation on S?

Clearly reflexive and symmetric. Is it transitive?

a) Yes, I can give a proof.
b) Yes, I think so, but I can't prove it.
c) No, I can give a proof.
d) No, I don't think so, but I can't prove it.

Equivalence Classes

Example:
Back to coins in bags.

Definition: Let R be an equivalence relation on S. The *equivalence class* of a ∈ S, $[a]_R$, is
$[a]_R$ = {b: aRb}

a is just a name for the equiv class. Any member of the class is a representative.

Equivalence Classes

What equivalence relation we've seen recently has representatives [244], [7], [58], [1]?

Equivalence Classes

Definition: Let R be an equivalence relation on S. The *equivalence class* of $a \in S$, $[a]_R$, is

$[a]_R = \{b: aRb\}$

Notice this is just a subset of S.

What does the set of equivalence classes on S look like?

To answer, think about the relation from before:

S = {people in this room}

R = {(a,b) : a has the same # of coins in his/her bag as b}

In how many different equivalence classes can each person fall?

Equivalence Classes

Lemma: Let R be an equivalence relation on S. Then

1. If aRb, then $[a]_R = [b]_R$
2. If not aRb, then $[a]_R \cap [b]_R = \emptyset$

Proof:

1. Suppose aRb, and consider $x \in S$.

$x \in [a]_R \Leftrightarrow aRx$ — Defn of $[a]_R$

$\Leftrightarrow xRa$ — symmetry

$\Leftrightarrow xRb$ — transitivity

$\Leftrightarrow bRx$ — symmetry

$\Leftrightarrow x \in [b]_R$ — Defn of $[b]_R$

Equivalence Classes

Lemma: Let R be an equivalence relation on S. Then

1. If aRb, then $[a]_R = [b]_R$
2. If not aRb, then $[a]_R \cap [b]_R = \emptyset$

Proof:

2. Suppose to the contrary that $\exists\, x \in [a]_R \cap [b]_R$.

$x \in [a]_R\ x \in [b]_R \Leftrightarrow aRx$ and bRx

$\Leftrightarrow aRx$ and xRb

$\Leftrightarrow aRb$, contradicting "not aRb"

$$S = \bigcup_{a \in S} [a]_R$$

Thus, $[a]_R$ and $[b]_R$ are either identical or disjoint.

Equivalence Classes

So S is the union of disjoint equivalence classes of R.

A partition of a set S is a (perhaps infinite…or uncountably infinite) collection of sets $\{A_i\}$ with

- Each A_i non-empty
- Each $A_i \subseteq S$
- For all i, j, $A_i \cap A_j = \varnothing$
- $S = \cup A_i$

Each A_i is called a block of the partition.

Equivalence Classes

Give me a partition of the reals into 2 blocks:

Give me a partition of the reals into 5 blocks:

Equivalence Classes

Theorem: if R is a _____ S, then $\{[a]_R : a \in S\}$ is a _____ S.

A. Partition of, equivalence relation on
B. Subset of, equivalence class of
C. Relation on, partition of
D. Equivalence relation on, partition of
E. I have no clue.

Theorem: if R is an equivalence relation on S, then $\{[a]_R : a \in S\}$ is a partition of S.

Proof: we need to show that an equivalence relation R satisfies the definition of a partition. (we've spent the whole day doing this!)

Equivalence Classes

Theorem: if {Ai} is *any* partition of S, then there exists an equivalence relation R, whose equivalence classes are exactly the blocks Ai.

Proof If {Ai} partitions S then define relation R on S to be $R = \{(a,b) : \exists\, i,\ a \in A_i \text{ and } b \in A_i\}$

Next show that R is an equivalence relation.

Reflexive and symmetric. Transitive?

Suppose aRb and bRc. Then a and b are in A_i, and b and c are in A_j.

But $b \in A_i \cap A_j$, so $A_i = A_j$.

So, a, b, $c \in A_i$, thus aRc.

Partially Ordered Sets (POSets)

Let R be a relation then R is a Partially Ordered Set (POSet) if it is

- Reflexive - aRa, ∀a
- Transitive - aRb ∧ bRc → aRc, ∀a,b,c
- *Anti*symmetric - aRb ∧ bRa → a=b, ∀a,b

Ex. (**R**,≤), the relation "≤" on the real numbers, is a partial order.

How do you check?

= Reflexive? a ≤ a for any real

order Transitive? If a ≤ b, b ≤ c then a ≤ c

< Antisymmetric? If a ≤ b, b ≤ a then a = b

Actually total order.

Partially Ordered Sets (POSets)

Ex. (Z^+, |), the relation "divides" on positive integers.

Reflexive? Yes, x|x since x=1x (k=1)

Transitive? Yes, or No?

a|b means b=ak, b|c means c=bj.
Does c=am for some m?

c = bj = akj (m=kj)

Antisymmetric?

a|b means b=ak, b|a means a=bj.
But b = bjk (subst) only if jk=1.

jk=1 means j=k=1, and we have b=a1, or b=a

Partially Ordered Sets (POSets)

Ex. (Z, |), the relation "divides" on integers. Not a poset.

Reflexive? Yes, x|x since x=1x (k=1)

Transitive?

a|b means b=ak, b|c means c=bj.
Does c=am for some m?

c = bj = akj (m=kj)

Antisymmetric? 3|-3, and -3|3, but 3 ≠ -3.

Yes, or No?

Partially Ordered Sets (POSets)

Ex. (2^S, ⊆), the relation "subset" on set of all subsets of S. A poset.

Reflexive? Yes, A ⊆ A, ∀A∈ 2^S

Transitive? A ⊆ B, B ⊆ C. Does that mean A ⊆ C?

A ⊆ B means x ∈A → x ∈B

B ⊆ C means x ∈B → x ∈C

Now take an x, and suppose it's in A. Must it also be in C? Yes, by MP

A. Modus Ponens
B. Modus Tollens
C. DeMorgan's
D. Transitivity

A ⊆ B, B ⊆ A → A=B

Partially Ordered Sets (POSets)

When we don't have a special relation definition in mind, we use the symbol "≤" to denote a partial order on a set.

When we know we're talking about a partial order, we'll write "a ≤ b" instead of "aRb" when discussing members of the relation.

We will also write "a < b" if a ≤ b and a ≠ b.

Partially Ordered Sets (POSets)

Ex. A common partial order on bit strings of length n, $\{0,1\}^n$, is defined as:

$a_1a_2\ldots a_n \leq b_1b_2\ldots b_n$

If and only if $a_i \leq b_i$, ∀ i.

A. 0110 ≤ 1000
B. 0110 ≤ 0000
C. 0110 ≤ 1110
D. 0110 ≤ 10111

0110 and 1000 are "incomparable" … We can't tell which is "bigger."

Actually, this relation is exactly the same as the last example, $(2^S, \subseteq)$.

In the string relation, we said 00≤01 because every bit in 00 is less than or = the corresp bit in 01.

String on the right has at least all the 1 bits of the left, maybe more. If each 1 represents an element in S, then right side has all elts of the left, maybe more.

Partially Ordered Sets (POSets)

Let (S, ≤) be a PO. If a ≤ b, or b ≤ a, then a and b are *comparable*. Otherwise, they are *incomparable*.

Ex. In poset $(Z^+, |)$, 3 and 6 are comparable, 6 and 3 are comparable, 3 and 5 are not, 8 and 12 are not.

A *total order* is a partial order where every pair of elements is comparable.

Ex. $(Z^+, \leq)$, is a total order, because for every pair (a,b) in ZxZ, either a ≤ b, or b ≤ a.

Dictionary order, or alphabetic order, or lexicographic order is a partial order on words in the english language. This idea can be generalized to strings over any alphabet.

Hasse Diagrams

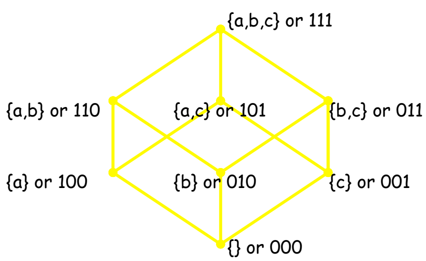
Hasse diagrams are a special kind of graphs used to describe posets.

Ex. In poset ({1,2,3,4}, ≤), we can draw the following picture to describe the relation.

1. Draw edge (a,b) if a ≤b
2. Don't draw up arrows
3. Don't draw self loops
4. Don't draw transitive edges

Hasse Diagrams

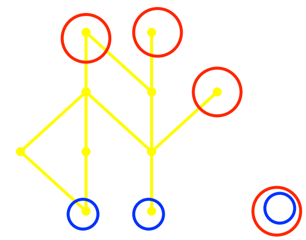
Have you seen this one before?



Hasse Diagrams

Consider this poset:

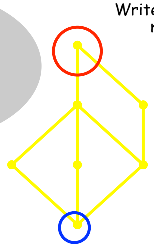
Reds are *maximal.*
Blues are *minimal.*



Hasse Diagrams

Definition: In a poset S, an element z is a *minimum element* if $\forall b \in S$, $z \leq b$.

Intuition: If a is maxiMAL, then no one beats a. If a is maxiMUM, a beats everything.



Write the defn of maximum!

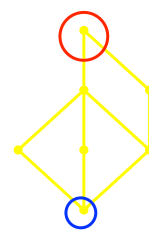
Did you get it right?

Must minimum and maximum exist?

A. Only if set is finite.
B. No.
C. Only if set is transitive.
D. Yes.

Hasse Diagrams

Theorem: In every poset, if the maximum element exists, it is unique. Similarly for minimum.

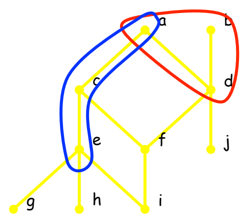


Proof: Suppose there are two maximum elements, a_1 and a_2, with $a_1 \neq a_2$.
Then $a_1 \leq a_2$, and $a_2 \leq a_1$, by defn of maximum.
So $a_1 = a_2$, a contradiction.
Thus, our supposition was incorrect, and the maximum element, if it exists, is unique.
Similar proof for minimum.

Upper and Lower Bounds

Defn: Let (S, ≤) be a partial order. If A⊆S, then an *upper bound* for A is any element x ∈ S (perhaps in A also) such that ∀ a ∈ A, a ≤ x.

A *lower bound* for A is any x ∈ S such that ∀ a ∈ A, a ≥ x.

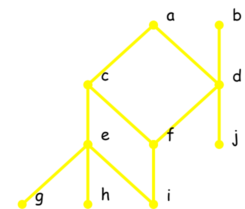


Ex. The upper bound of {g,j} is a. Why not b?

Upper and Lower Bounds

Defn: Let (S, ≤) be a partial order. If A⊆S, then an *upper bound* for A is any element x ∈ S (perhaps in A also) such that ∀ a ∈ A, a ≤ x.

A *lower bound* for A is any x ∈ S such that ∀ a ∈ A, a ≥ x.



Ex. The upper bound of {g,j} is a. Why not b?

Ex. The upper bounds of {g,i} is/are…

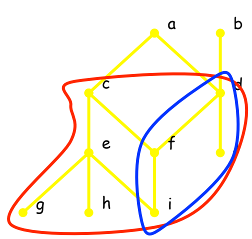
A. I have no clue.
B. c and e
C. a
D. a, c, and e

{a, b} has no UB.

Upper and Lower Bounds

Defn: Let (S, ≤) be a partial order. If A⊆S, then an *upper bound* for A is any element x ∈ S (perhaps in A also) such that ∀ a ∈ A, a ≤ x.

A *lower bound* for A is any x ∈ S such that ∀ a ∈ A, a ≥ x.



Ex. The lower bounds of {a,b} are d, f, i, and j.

Ex. The lower bounds of {c,d} is/are…

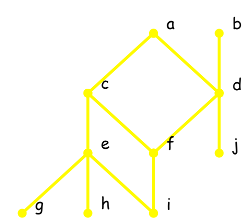
A. I have no clue.
B. f, i
C. j, i, g, h
D. e, f, j

{g, h, i, j} has no LB.

Upper and Lower Bounds

Defn: Given poset (S, ≤) and A⊆S, x ∈ S is a *least upper bound* (LUB) for A if x is an upper bound and for upper bound y of A, y ≥ x.

x is a *greatest lower bound* for A if x is a lower bound and if x ≤ y for every UB y of A.



Ex. LUB of {i,j} = d.

Ex. GLB of {g,j} is…

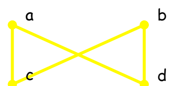
A. I have no clue.
B. a
C. non-existent
D. e, f, j

Upper and Lower Bounds

This is because c and d are incomparable.

Ex. In the following poset, c and d are lower bounds for {a, there is no GLB.

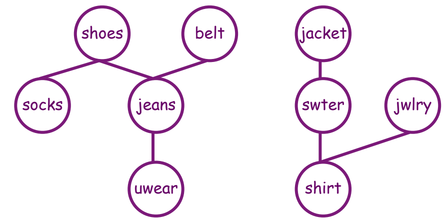
Similarly, a and b are upper bounds for {c,d}, but there is no LUB.



Total Orders (Epp p. 639)

Consider the problem of getting dressed.

Precedence constraints are modeled by a poset in which a ≤ b iff you must put on a before b.



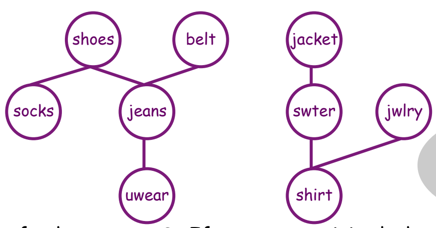
In what order will you get dressed while respecting constraints?

Let (S, ≤) be a poset (S finite). We will extend ≤ to a total order on S, so we can decide for all incomparable pairs whether to make a ≤ b, or vice versa w/o violating T,R,A.

Total Orders

Things we need:

Lemma: Every finite non-empty poset (S, ≤) has at least one minimal element.



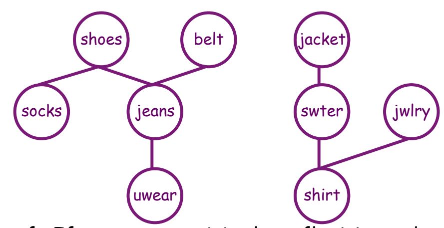
NOT A REAL PROOF! Should use induction!!

Proof: choose $a_0 \in S$. If a_0 was not minimal, then there exists $a_1 \le a_0$, and so on until a minimal element is found.

Total Orders

More things we need:

Lemma: If (S, ≤) is a poset with a minimal, then (S-{a}, ≤) is also a poset.



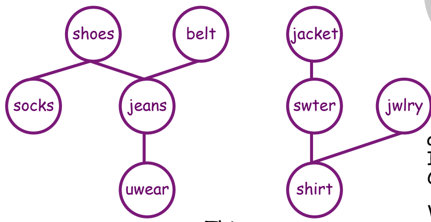
Proof: If you remove minimal a reflexivity and antisymmetry still hold. If x,y,z ∈ S-{a}, with x ≤ y and y ≤ z, then x ≤ z too, since (S, ≤) was transitive.

Total Orders (Epp p. 641)

Think about what this means:

1. There is always a minimal element.
2. If you remove it you still have a poset.

Depending on which min elt is chosen each time, a different total order is obtained, but all TOs will be consistent with the PO.



This suggests:

alg Topological Sort
Input: poset $(S, \leq)$
Out: elements of S in total order
While $S \neq \emptyset$
Remove any min elt from S and output it.

Total Orders

alg Topological Sort
Input: poset $(S, \leq)$
Out: elements of S in total order
While $S \neq \emptyset$
Remove any min elt from S and output it.