Degrees of Freedom of the Two-way Interference Channel with a Non causal Multi-antenna Relay

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***Abstract*—We characterize the degrees of freedom (DoF) of the full-duplex two-way interference channel with a non causal multi-antenna relay and time-varying channels. In a two-way interference channel (IC), 4 messages and 4 transmitters/receivers form an IC in the forward direction (2 messages) and another IC in the backward direction (2 messages) which operate simultaneously in full-duplex mode. Furthermore, all nodes are permitted to interact, i.e. adapt current channel inputs to past received signals. We propose a novel block Markov coding scheme combining the ideas of interference alignment and successive decoding to achieve the full, maximal, 4 degrees of freedom asymptotically. All source/destination nodes have a single antenna while 4 antennas at the relay are sufficient to achieve the full DoF. Interestingly, this implies that the non causal relay is able to effectively mitigate interference in this two way setting – i.e. each user in the two-way interference channel is able to exchange information with its desired user at interference-free rates thanks to the relay.**

I. Introduction

In wireless communications, current two-way systems typically employ either time or frequency division to achieve two-way or bidirectional communication. This restriction is due to a combination of hardware and implementation imperfections and effectively orthogonalizes the two directions, rendering the bidirectional channel equivalent to two one-way communication systems. However, recently much progress has been made on the design of full-duplex (in-band) wireless systems [1], [2], which show great promise for increasing data rates in future wireless technologies. In this work we seek to understand the potential of full duplex systems in a multi-user or network setting, and do so from a multi-user information theoretic perspective.

We consider the two-way interference channel [3]–[6], in which there are 4 independent messages: two-messages to be transmitted over an interference channel (IC) in the $\rightarrow$ direction simultaneously with two-messages to be transmitted over an in-band IC in the $\leftarrow$ direction. The degrees of freedom (DoF) [7], an approximate capacity characterization that intuitively corresponds to the number of independent interference-free signals that can be communicated in a network, of the two-way interference channel have been shown to be 2 [3] – i.e. 1 in each direction.

Relays are additional nodes which do not have messages of their own and may aid the other nodes in transmitting their signals. In this work we ask how much a relay may increase the DoF of the full duplex two-way interference channel. Very interestingly, we show that while the DoF of the two-way IC is 2 – indicating that interference is present in the IC in each direction – that the presence of a non causal multi-antenna relay may increase the DoF to the maximal value of 4. That is, in a DoF sense, a non causal multi-antenna relay may somehow cancel out all the interference in the two-way IC and each user in the network is able to communicate with its desired user in a completely interference-free environment.

The degrees of freedom of a variety of one-way communication networks have been characterized [8]–[12]. However, much less is known about the DoF of two-way communications. Very recently, [13] considered a *half-duplex* two-pair two-way interference channel (where nodes other than the relay may not employ interaction, i.e. transmit signals are functions of the messages only and not past outputs) with a 2-antenna relay and showed that 4/3 DoF is achievable. No converse results where provided. In [14], the authors identified the DoF of the full-duplex 2-pair and 3-pair two-way multi-antenna relay MIMO interference channel, in which there is no interference between users who only communicate through the relay. The DoF per user was shown to be piecewise linear depending on the number of antennas at the users and relay.

In this work, we propose a novel block Markov coding scheme based on the combined ideas of interference alignment [15]–[17] (at the destinations) and successive decoding (at the relay) for the *full-duplex* two-way interference channel with a non causal multi-antenna relay. All nodes may employ interaction – i.e. signals may be a function of previously received outputs. The main message is that, with 4 antennas, the relay may intelligently manage the interference seen at all 4 receivers *simultaneously* and effectively mitigate it to achieve the optimal 4 DoF. This is based on using the relay to enable aligning, decoding, and subtracting the interference from received signals as a whole (in our scheme by decoding the sum of signals for example), in addition to zero-forcing beam forming enabled by the multiple antennas.

The remainder of the paper is organized as follows: We mathematically model our system in Section II, then propose a block Markov coding scheme to achieve 8/3 DoF as a motivating example in Section III. The main results of the paper and a few remarks are presented in Section IV. We conclude the paper in Section V.

II. System Model

We consider a two-way interference channel with a multi-antenna relay as shown in Fig. 1. The two-way IC has 4 messages and 4 terminals forming an IC in the $\rightarrow$ direction

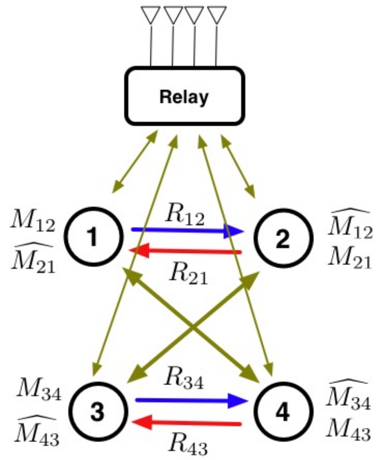

Fig. 1. Full-duplex two-way interference channel with a multi-antenna relay.

(2 messages) and another IC in the $\leftarrow$ direction (2 messages). There is a multi-antenna relay which helps in communicating messages and managing interference in the network. All nodes are able to operate in full-duplex mode, i.e. they can transmit and receive signals simultaneously.

The relay is assumed to have 4 antennas and to operate in a non causal fashion. By "non causal" we refer to its ability to decode and forward signals received at the previous and *current* time slots. We note that this requirement is significantly less strict than a *cognitive* relay, which would know all users' signals prior to transmission and does not obtain the messages over the air. Here messages are obtained over the air; the only idealization is the non causality. Mathematically, we may describe this non causal relay function as follows

$$\mathbf{X_R}[k] = g(\mathbf{Y_R}[1], \mathbf{Y_R}[2], ..., \mathbf{Y_R}[k])$$

where $\mathbf{X_R}[k]$ is a 4×1 (4 antennas) vector signal transmitted by the relay at time slot k; $g()$ is a deterministic function; and $\mathbf{Y_R}[l], l \in \{1, 2, ..., k\}$ are the received 4×1 vector signals at the relay at time slot l.

At each time slot k, the system input/output relationships are described as:

$$Y_1[k] = h_{21}[k]X_2[k] + h_{41}[k]X_4[k] + \mathbf{h}^*_{\mathbf{R1}}[k]\mathbf{X_R}[k] + Z_1[k] \quad (1)$$

$$Y_2[k] = h_{12}[k]X_1[k] + h_{32}[k]X_3[k] + \mathbf{h}^*_{\mathbf{R2}}[k]\mathbf{X_R}[k] + Z_2[k] \quad (2)$$

$$Y_3[k] = h_{23}[k]X_2[k] + h_{43}[k]X_4[k] + \mathbf{h}^*_{\mathbf{R3}}[k]\mathbf{X_R}[k] + Z_3[k] \quad (3)$$

$$Y_4[k] = h_{14}[k]X_1[k] + h_{34}[k]X_3[k] + \mathbf{h}^*_{\mathbf{R4}}[k]\mathbf{X_R}[k] + Z_4[k] \quad (4)$$

$$\mathbf{Y_R}[k] = \mathbf{h_{1R}}[k]X_1[k] + \mathbf{h_{2R}}[k]X_2[k] + \mathbf{h_{3R}}[k]X_3[k] + \mathbf{h_{4R}}[k]X_4[k] + \mathbf{Z_R}[k] \quad (5)$$

where $X_i[k], Y_i[k], i \in \{1, 2, 3, 4\}$ are the inputs and outputs of user i at time slot k, and $h_{ij}[k](\mathbf{h_{ij}}[k]), i, j \in \{1, 2, 3, 4, R\}$ is the channel coefficient (vector) from node i to node j at time slot k. The terms in bold represent vectors due to the multi-antenna relay. We use $*$ to denote conjugate transpose. The network is subject to Gaussian noise $Z_i[k] \sim \mathcal{CN}(0, 1), i \in \{1, 2, 3, 4\}$ and $\mathbf{Z_R}[k] \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$ (which we will be ignoring as DoF is an asymptotic measure where the signal to noise ratio $\to \infty$). We consider time-varying channel coefficients, which are all drawn from a continuous distribution and whose absolute values are bounded between a nonzero minimum value and a finite maximum value. Note one can also alternatively consider a frequency selective system model.

Let P be the transmit power constraint at each user and the relay: $E[|X_i[k]|^2] \leq P, i \in \{1, 2, 3, 4\}$ and $E[||\mathbf{X_R}[k]||_2^2] \leq P$. The channel state information is assumed to be perfectly known at the nodes in receiving mode and the relay has global channel state information (CSI) knowledge for all links.

User i sends an independent message M_{ij} for one intended user j (user 1 sends to 2, 2 to 1, 3 to 4, and 4 to 3) with an *interactive* encoding function $X_i[k] = f(M_{ij}, Y_i^{[k-1]})$ at rate $R_{ij}(P) = \frac{\log_2 |M_{ij}|}{n}$, where $Y_i^{[k-1]}$ denotes previous signals (from time slot 1 to $k-1$) received at user i. In other words, all users in this network can adapt current channel inputs to previously received channel outputs. A rate tuple $(R_{12}, R_{21}, R_{34}, R_{43})$ is said to be achievable if each receiver can decode the desired message with arbitrarily small probability of error when the number of channel uses n tends to infinity. Then, the sum degrees of freedom characterizing the approximate sum capacity at high SNR is defined as the maximum over all achievable $(R_{12}, R_{21}, R_{34}, R_{43})$ of

$$\begin{aligned} d_{sum} &= d_{12} + d_{21} + d_{34} + d_{43} \\ &= \limsup_{P \to \infty} \frac{R_{12}(P) + R_{21}(P) + R_{34}(P) + R_{43}(P)}{\log(P)}. \end{aligned}$$

Let $s^1_{ij}, s^2_{ij}, s^3_{ij}, ..., s^k_{ij}$ denote the information symbols (signals) sent by transmitter i to receiver j. Note the signals are not necessarily independent due to the interactive encoding function $X_i[k] = f(M_{ij}, Y_i^{[k-1]})$. However, we will show that a non-interactive achievability scheme suffices to achieve the full DoF where interactive encoding functions are permitted in the converse.

The received signal may be broken down into four types of signals: the self-interference signal (SI, sent by itself, known to itself), the interference signal (sent by the undesired user from the opposite side), the desired signal (sent by the desired user) and the undesired signal (sent by the undesired user from the same side) respectively. For example, at receiver 1, $s^l_{12}, l \in \{1, 2, ..., k\}$ are self-interference signals (SI); $s^l_{43}, l \in \{1, 2, ..., k\}$ are interference signals; $s^l_{21}, l \in \{1, 2, ..., k\}$ are desired signals and $s^l_{34}, l \in \{1, 2, ..., k\}$ are undesired signals.

Note we have already removed self-interference signals from the input/output equations (1)-(4), but SI terms may still be transmitted by the relay and hence received.

III. A MOTIVATING EXAMPLE

In this section, as a motivating example, we present a block Markov coding scheme to achieve 8 DoF in 3 time slots (or a sum DoF of 8/3) for the two-way interference channel with the help of a non causal 4-antenna relay. As there are only 3 time slots, notation is simpler to understand and this is meant to be an intuitive and illustrative example; to achieve the full 4 DoF the scheme presented for the 3 slot case naturally generalizes but requires an asymptotically large number of time slots (presented in the subsequent section). We consider Gaussian channel model at high SNR and noise terms are ignored from now on.

At time slot 1, each transmitter sends the sum of two new symbols, i.e. $s_{ij}^1 + s_{ij}^2$, and the relay listens. At the receivers,

$$Y_1[1] = h_{21}[1](s_{21}^1 + s_{21}^2) + h_{41}[1](s_{43}^1 + s_{43}^2) \quad (6)$$

$$Y_2[1] = h_{12}[1](s_{12}^1 + s_{12}^2) + h_{32}[1](s_{34}^1 + s_{34}^2) \quad (7)$$

$$Y_3[1] = h_{43}[1](s_{43}^1 + s_{43}^2) + h_{23}[1](s_{21}^1 + s_{21}^2) \quad (8)$$

$$Y_4[1] = h_{34}[1](s_{34}^1 + s_{34}^2) + h_{14}[1](s_{12}^1 + s_{12}^2) \quad (9)$$

$$\begin{aligned}\mathbf{Y_R}[1] &= \mathbf{h_{1R}}[1](s_{12}^1 + s_{12}^2) + \mathbf{h_{2R}}[1](s_{21}^1 + s_{21}^2) \\ &\quad + \mathbf{h_{3R}}[1](s_{34}^1 + s_{34}^2) + \mathbf{h_{4R}}[1](s_{43}^1 + s_{43}^2). \end{aligned} \quad (10)$$

The 4-antenna relay is able (global CSI) to decode 4 sums of 2 messages $s_{ij}^1 + s_{ij}^2$ using a zero-forcing decoder.

At time slot 2, each transmitter sends another sum of two symbols, i.e. $s_{ij}^1 - s_{ij}^2$. The relay receives

$$\begin{aligned}\mathbf{Y_R}[2] &= \mathbf{h_{1R}}[2](s_{12}^1 - s_{12}^2) + \mathbf{h_{2R}}[2](s_{21}^1 - s_{21}^2) \\ &\quad + \mathbf{h_{3R}}[2](s_{34}^1 - s_{34}^2) + \mathbf{h_{4R}}[2](s_{43}^1 - s_{43}^2).\end{aligned}$$

and decodes $s_{ij}^1 - s_{ij}^2$ at time slot 2. The non causal relay combines this with $s_{ij}^1 + s_{ij}^2$ from time slot 1 and at slot 2 is thus able to decode all 8 messages and transmits:

$$\begin{aligned}\mathbf{X_R}[2] &= \mathbf{u_{12}}[2]s_{12}^1 + \mathbf{v_{12}}[2]s_{12}^2 + \mathbf{u_{21}}[2]s_{21}^1 + \mathbf{v_{21}}[2]s_{21}^2 \\ &\quad + \mathbf{u_{34}}[2]s_{34}^1 + \mathbf{v_{34}}[2]s_{34}^2 + \mathbf{u_{43}}[2]s_{43}^1 + \mathbf{v_{43}}[2]s_{43}^2,\end{aligned}$$

where $\mathbf{u_{ij}}[2]$ and $\mathbf{v_{ij}}[2]$ denote the 4×1 beamforming vectors carrying signals from user i to user j at time slot 2. To prevent undesired signals from reaching the receivers, the relay picks the specific beamforming vectors such that

$$\mathbf{u_{34}}[2], \mathbf{v_{34}}[2] \in null(\mathbf{h_{R1}^*}[2]), \quad (11)$$

$$\mathbf{u_{43}}[2], \mathbf{v_{43}}[2] \in null(\mathbf{h_{R2}^*}[2]), \quad (12)$$

$$\mathbf{u_{12}}[2], \mathbf{v_{12}}[2] \in null(\mathbf{h_{R3}^*}[2]), \quad (13)$$

$$\mathbf{u_{21}}[2], \mathbf{v_{21}}[2] \in null(\mathbf{h_{R4}^*}[2]), \quad (14)$$

where $null(\mathbf{A})$ denotes the null space of $\mathbf{A}$. Note that $null(\mathbf{h_{Rj}^*}[2]), j \in \{1, 2, 3, 4\}$ is of dimension 3. Now, at the receiver side of users, we have equations (15)-(18):

$$\begin{aligned}Y_1[2] &= h_{21}[2](s_{21}^1 - s_{21}^2) + h_{41}[2](s_{43}^1 - s_{43}^2) + \mathbf{h_{R1}^*}[2]\mathbf{u_{21}}[2]s_{21}^1 \\ &+ \mathbf{h_{R1}^*}[2]\mathbf{v_{21}}[2]s_{21}^2 + \mathbf{h_{R1}^*}[2]\mathbf{u_{43}}[2]s_{43}^1 + \mathbf{h_{R1}^*}[2]\mathbf{v_{43}}[2]s_{43}^2 \\ &+ \mathbf{h_{R1}^*}[2]\mathbf{u_{12}}[2]s_{12}^1 + \mathbf{h_{R1}^*}[2]\mathbf{v_{12}}[2]s_{12}^2\end{aligned} \quad (15)$$

$$\begin{aligned}Y_2[2] &= h_{12}[2](s_{12}^1 - s_{12}^2) + h_{32}[2](s_{34}^1 - s_{34}^2) + \mathbf{h_{R2}^*}[2]\mathbf{u_{12}}[2]s_{12}^1 \\ &+ \mathbf{h_{R2}^*}[2]\mathbf{v_{12}}[2]s_{12}^2 + \mathbf{h_{R2}^*}[2]\mathbf{u_{34}}[2]s_{34}^1 + \mathbf{h_{R2}^*}[2]\mathbf{v_{34}}[2]s_{34}^2 \\ &+ \mathbf{h_{R2}^*}[2]\mathbf{u_{21}}[2]s_{21}^1 + \mathbf{h_{R2}^*}[2]\mathbf{v_{21}}[2]s_{21}^2\end{aligned} \quad (16)$$

$$\begin{aligned}Y_3[2] &= h_{43}[2](s_{43}^1 - s_{43}^2) + h_{23}[2](s_{21}^1 - s_{21}^2) + \mathbf{h_{R3}^*}[2]\mathbf{u_{43}}[2]s_{43}^1 \\ &+ \mathbf{h_{R3}^*}[2]\mathbf{v_{43}}[2]s_{43}^2 + \mathbf{h_{R3}^*}[2]\mathbf{u_{21}}[2]s_{21}^1 + \mathbf{h_{R3}^*}[2]\mathbf{v_{21}}[2]s_{21}^2 \\ &+ \mathbf{h_{R3}^*}[2]\mathbf{u_{34}}[2]s_{34}^1 + \mathbf{h_{R3}^*}[2]\mathbf{v_{34}}[2]s_{34}^2\end{aligned} \quad (17)$$

$$\begin{aligned}Y_4[2] &= h_{34}[2](s_{34}^1 - s_{34}^2) + h_{14}[2](s_{12}^1 - s_{12}^2) + \mathbf{h_{R4}^*}[2]\mathbf{u_{34}}[2]s_{34}^1 \\ &+ \mathbf{h_{R4}^*}[2]\mathbf{v_{34}}[2]s_{34}^2 + \mathbf{h_{R4}^*}[2]\mathbf{u_{12}}[2]s_{12}^1 + \mathbf{h_{R4}^*}[2]\mathbf{v_{12}}[2]s_{12}^2 \\ &+ \mathbf{h_{R4}^*}[2]\mathbf{u_{43}}[2]s_{43}^1 + \mathbf{h_{R4}^*}[2]\mathbf{v_{43}}[2]s_{43}^2\end{aligned} \quad (18)$$

To align interference signals at time slot 2 – i.e. have them be received with the same coefficients as the interference signals at time slot 1 as $s_{ij}^1 + s_{ij}^2$, we design beamforming vectors so that the following equations are satisfied:

$$\mathbf{h_{R1}^*}[2]\mathbf{v_{43}}[2] - \mathbf{h_{R1}^*}[2]\mathbf{u_{43}}[2] = 2h_{41}[2] \quad (19)$$

$$\mathbf{h_{R2}^*}[2]\mathbf{v_{34}}[2] - \mathbf{h_{R2}^*}[2]\mathbf{u_{34}}[2] = 2h_{32}[2] \quad (20)$$

$$\mathbf{h_{R3}^*}[2]\mathbf{v_{21}}[2] - \mathbf{h_{R3}^*}[2]\mathbf{u_{21}}[2] = 2h_{23}[2] \quad (21)$$

$$\mathbf{h_{R4}^*}[2]\mathbf{v_{12}}[2] - \mathbf{h_{R4}^*}[2]\mathbf{u_{12}}[2] = 2h_{14}[2]. \quad (22)$$

This will allow us to decode the interference "sums" $s_{ij}^1 + s_{ij}^2$ at each receiver and subtract this off.

Assuming the interference has been decoded and subtracted off (we will argue why this is possible later), to decode the desired signals after time slot 3, the following matrices in the first 2 time slots must be full-rank:

$$\begin{bmatrix} h_{21}[1] & h_{21}[1] \\ h_{21}[2] + \mathbf{h_{R1}^*}[2]\mathbf{u_{21}}[2] & -h_{21}[2] + \mathbf{h_{R1}^*}[2]\mathbf{v_{21}}[2] \end{bmatrix} \quad (23)$$

$$\begin{bmatrix} h_{12}[1] & h_{12}[1] \\ h_{12}[2] + \mathbf{h_{R2}^*}[2]\mathbf{u_{12}}[2] & -h_{12}[2] + \mathbf{h_{R2}^*}[2]\mathbf{v_{12}}[2] \end{bmatrix} \quad (24)$$

$$\begin{bmatrix} h_{43}[1] & h_{43}[1] \\ h_{43}[2] + \mathbf{h_{R3}^*}[2]\mathbf{u_{43}}[2] & -h_{43}[2] + \mathbf{h_{R3}^*}[2]\mathbf{v_{43}}[2] \end{bmatrix} \quad (25)$$

$$\begin{bmatrix} h_{34}[1] & h_{34}[1] \\ h_{34}[2] + \mathbf{h_{R4}^*}[2]\mathbf{u_{34}}[2] & -h_{34}[2] + \mathbf{h_{R4}^*}[2]\mathbf{v_{34}}[2] \end{bmatrix} \quad (26)$$

Since the beamforming vectors $\mathbf{u_{ij}}[2], \mathbf{v_{ij}}[2]$ were designed independently of $\mathbf{h_{Rj}^*}[2]$, and the channel coefficients $h_{ij}[1], h_{ij}[2]$ were drawn from a continuous random distribution, the 4 matrices above have full rank almost surely.

That such beam forming vectors satisfying the needed constraints always exist follows by a dimensionality argument (along with the random channel coefficients). Take $\mathbf{u_{34}}[2]$ and $\mathbf{v_{34}}[2]$ as an example. We want to find $\mathbf{u_{34}}[2]$ and $\mathbf{v_{34}}[2]$ such that (11) and (20) are satisfied. From (11) there are 6 free parameters, which are reduced to 5 in order to satisfy (20). Thus, let a, b, c, d, e be five scalars and let

$$\mathbf{v_{34}}[2] = \begin{bmatrix} \mathbf{h_{R1}^*}[2] \\ \mathbf{h_{R2}^*}[2] \\ \mathbf{A} \\ \mathbf{B} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 2h_{32}[2] + a \\ b \\ c \end{bmatrix} \quad (27)$$

$$\mathbf{u_{34}}[2] = \begin{bmatrix} \mathbf{h_{R1}^*}[2] \\ \mathbf{h_{R2}^*}[2] \\ \mathbf{A} \\ \mathbf{B} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ a \\ d \\ e \end{bmatrix} \quad (28)$$

where $\mathbf{A}$ and $\mathbf{B}$ are 1×4 matrices such that $[\mathbf{h_{R1}^*}[2]\ \mathbf{h_{R2}^*}[2]\ \mathbf{A}\ \mathbf{B}]^T$ has full rank. This is always possible as $\mathbf{h}^*_{\mathbf{R1}}[2], \mathbf{h}^*_{\mathbf{R2}}[2]$ are drawn from a continuous distribution and hence with high probability are not equal. This choice satisfies (11) and (20) while guaranteeing the matrix (26) full-rank almost surely (as they are not designed based on $\mathbf{h}^*_{\mathbf{R4}}$).

At time slot 3, all transmitters listen and the relay broadcasts 4 sums of symbols:

$$\mathbf{X_R}[3] = \mathbf{u_{12}}[3](s^1_{12} + s^2_{12}) + \mathbf{u_{21}}[3](s^1_{21} + s^2_{21}) + \mathbf{u_{34}}[3](s^1_{34} + s^2_{34}) + \mathbf{u_{43}}[3](s^1_{43} + s^2_{43})$$

Again, we need the following beamforming vectors to nullify the undesired signal at each receiver:

$$\mathbf{u_{34}}[3] \in null(\mathbf{h}^*_{\mathbf{R1}}[3]), \quad \mathbf{u_{43}}[3] \in null(\mathbf{h}^*_{\mathbf{R2}}[3])$$
$$\mathbf{u_{12}}[3] \in null(\mathbf{h}^*_{\mathbf{R3}}[3]), \quad \mathbf{u_{21}}[3] \in null(\mathbf{h}^*_{\mathbf{R4}}[3]).$$

Now, at the receiver side,

$$Y_1[3] = \mathbf{h}^*_{\mathbf{R1}}[3]\mathbf{u_{12}}[3](s^1_{12} + s^2_{12}) + \mathbf{h}^*_{\mathbf{R1}}[3]\mathbf{u_{21}}[3](s^1_{21} + s^2_{21}) + \mathbf{h}^*_{\mathbf{R1}}[3]\mathbf{u_{43}}[3](s^1_{43} + s^2_{43}) \quad (29)$$

$$Y_2[3] = \mathbf{h}^*_{\mathbf{R2}}[3]\mathbf{u_{12}}[3](s^1_{12} + s^2_{12}) + \mathbf{h}^*_{\mathbf{R2}}[3]\mathbf{u_{21}}[3](s^1_{21} + s^2_{21}) + \mathbf{h}^*_{\mathbf{R2}}[3]\mathbf{u_{34}}[3](s^1_{34} + s^2_{34}) \quad (30)$$

$$Y_3[3] = \mathbf{h}^*_{\mathbf{R3}}[3]\mathbf{u_{34}}[3](s^1_{34} + s^2_{34}) + \mathbf{h}^*_{\mathbf{R3}}[3]\mathbf{u_{21}}[3](s^1_{21} + s^2_{21}) + \mathbf{h}^*_{\mathbf{R3}}[3]\mathbf{u_{43}}[3](s^1_{43} + s^2_{43}) \quad (31)$$

$$Y_4[3] = \mathbf{h}^*_{\mathbf{R4}}[3]\mathbf{u_{12}}[3](s^1_{12} + s^2_{12}) + \mathbf{h}^*_{\mathbf{R4}}[3]\mathbf{u_{34}}[3](s^1_{34} + s^2_{34}) + \mathbf{h}^*_{\mathbf{R4}}[3]\mathbf{u_{43}}[3](s^1_{43} + s^2_{43}) \quad (32)$$

To decode the sum of interference signals at each receiver by using (6)-(9) and (29)-(32), the following must be full-rank:

$$\begin{bmatrix} h_{21}[1] & h_{41}[1] \\ \mathbf{h}^*_{\mathbf{R1}}[3]\mathbf{u_{21}}[3] & \mathbf{h}^*_{\mathbf{R1}}[3]\mathbf{u_{43}}[3] \end{bmatrix}, \begin{bmatrix} h_{12}[1] & h_{32}[1] \\ \mathbf{h}^*_{\mathbf{R2}}[3]\mathbf{u_{12}}[3] & \mathbf{h}^*_{\mathbf{R2}}[3]\mathbf{u_{34}}[3] \end{bmatrix}$$
$$\begin{bmatrix} h_{43}[1] & h_{23}[1] \\ \mathbf{h}^*_{\mathbf{R3}}[3]\mathbf{u_{43}}[3] & \mathbf{h}^*_{\mathbf{R3}}[3]\mathbf{u_{21}}[3] \end{bmatrix}, \begin{bmatrix} h_{34}[1] & h_{14}[1] \\ \mathbf{h}^*_{\mathbf{R4}}[3]\mathbf{u_{34}}[3] & \mathbf{h}^*_{\mathbf{R4}}[3]\mathbf{u_{12}}[3] \end{bmatrix}.$$

Since the beamforming vectors $\mathbf{u_{ij}}[3]$ were designed independently of $\mathbf{h}^*_{\mathbf{Rj}}[3]$, and the channel coefficients $h_{ij}[1]$ were drawn from a continuous random distribution, the 4 matrices above have full rank almost surely.

Decoding: Now each receiver is able to decode 2 desired signals and the sum of interference signals. For example, at receiver 1, combining equation (6), (15) and (29), we are able to decode s^1_{21}, s^2_{21} and $s^1_{43} + s^2_{43}$ as follows:

• Subtract SI signals $s^1_{12} + s^2_{12}$ in (29), then combine with (6) to decode the sum of interference signals $s^1_{43} + s^2_{43}$.

• Rewrite (15) as follows since we have chosen the beamforming vectors to align interference signals in (19):

$$Y_1[2] = (h_{21}[2] + \mathbf{h}^*_{\mathbf{R1}}[2]\mathbf{u_{21}}[2])s^1_{21} + (-h_{21}[2] + \mathbf{h}^*_{\mathbf{R1}}[2]\mathbf{v_{21}}[2])s^2_{21} + (h_{41}[2] + \mathbf{h}^*_{\mathbf{R1}}[2]\mathbf{u_{43}}[2])(s^1_{43} + s^2_{43}) + \mathbf{h}^*_{\mathbf{R1}}[2]\mathbf{u_{12}}[2]s^1_{12} + \mathbf{h}^*_{\mathbf{R1}}[2]\mathbf{v_{12}}[2]s^2_{12} \quad (33)$$

• Subtract the sum of interference signals in equation (6) and (33), as well as SI signals in (33). Decode the desired signals s^1_{21}, s^2_{21} using (6) and (33).

Similar decoding procedures follow for the other 3 receivers. Therefore, after 3 time slots, the proposed block Markov coding scheme achieves 8/3 sum DoF.

Remark 1: In the proposed scheme, receivers want to decode all desired signals individually and align all interference signals from the undesired transmitter so that it can decode this aligned interference (in this case the sum). The non causal 4-antenna relay simultaneously helps *each* receiver to align the interference signals and nullify the other undesired signal through beamforming. By following the same procedure, it is possible to achieve 12 sum DoF in 4 time slots, 16 sum DoF in 5 time slots, 20 sum DoF in 6 time slots, and in general, to achieve $4k$ sum DoF in $k + 1$ time slots, or a DoF of $\frac{4k}{k+1}, k \geq 2, k \in \mathbb{N}$. Therefore, as $k \to \infty$, 4 DoF is achievable for the full-duplex two-way IC with a non causal multi-antenna relay, as described next.

IV. Degrees of freedom of two-way IC with a non causal 4-antenna relay

In this section, we propose a block Markov coding scheme that achieves 4 DoF for the two-way IC with the help of a non causal 4-antenna relay. The main technical result of the paper is stated in the following theorem, whose proof is given in the remainder of this section.

Theorem 1: The full-duplex two-way interference channel with a non causal 4-antenna relay has 4 degrees of freedom.

A. Converse

The converse of Theorem 1 is trivial since for a 4-user, 4 message unicast network where all sources and destinations have a single antenna, even if we consider interactive encoding functions $X_i[k] = f(M_{ij}, Y_i^{[k-1]})$, the maximum degrees of freedom cannot exceed 4 by cut-set arguments.

B. Achievability

We propose a novel block Markov coding scheme to achieve $4k$ DoF in $k + 1$ time slots. The main ideas used are interference alignment and successive decoding by the relay.

Each user transmits a linear combination of k symbols in each time slot except the last time slot as in Table I.

At the relay, it hears all signals from all transmitters at each time slot except the last time slot and at each time slot is able to decode 4 linear combinations of messages using a zero-forcing decoder (all $\mathbf{h}_{iR}$ are different with high probability). Combining these 4 combinations decoded at the current time slot with those decoded in the previous time slot(s), at each time slot 4 new individual messages are able to be decoded. We term this successive decoding of all the individual messages at the relay (over successive time slots); what the relay know at each time slot is presented in Table II.

Given this knowledge, from time slot 2 onwards, the relay broadcasts the signals as in Table III. We use $\mathbf{u_{ij}}[k], \mathbf{v_{ij}}[k]$ to denote beamforming vectors carrying signals from transmitter i to receiver j at time slot k.

Time Slot	User 1	User 2	User 3	User 4
1	$s_{12}^1 + s_{12}^2 + ... + s_{12}^k$	$s_{21}^1 + s_{21}^2 + ... + s_{21}^k$	$s_{34}^1 + s_{34}^2 + ... + s_{34}^k$	$s_{43}^1 + s_{43}^2 + ... + s_{43}^k$
2	$s_{12}^1 + s_{12}^2 + ... - s_{12}^k$	$s_{21}^1 + s_{21}^2 + ... - s_{21}^k$	$s_{34}^1 + s_{34}^2 + ... - s_{34}^k$	$s_{43}^1 + s_{43}^2 + ... - s_{43}^k$
3	$s_{12}^1 + ... - s_{12}^{k-1} + s_{12}^k$	$s_{21}^1 + ... - s_{21}^{k-1} + s_{21}^k$	$s_{34}^1 + ... - s_{34}^{k-1} + s_{34}^k$	$s_{43}^1 + ... - s_{43}^{k-1} + s_{43}^k$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	$s_{12}^1 - s_{12}^2 + ... + s_{12}^k$	$s_{21}^1 - s_{21}^2 + ... + s_{21}^k$	$s_{34}^1 - s_{34}^2 + ... + s_{34}^k$	$s_{43}^1 - s_{43}^2 + ... + s_{43}^k$
k+1	none	none	none	none

TABLE I

SIGNALS TRANSMITTED BY 4 USERS IN $k+1$ TIME SLOTS.

Time Slot	Signal Sent by Relay
1	none
2	$\mathbf{u_{12}}[2](s_{12}^1 + s_{12}^2 + ... + s_{12}^{k-1}) + \mathbf{v_{12}}[2]s_{12}^k + \mathbf{u_{21}}[2](s_{21}^1 + s_{21}^2 + ... + s_{21}^{k-1}) + \mathbf{v_{21}}[2]s_{21}^k$
	$+\mathbf{u_{34}}[2](s_{34}^1 + s_{34}^2 + ... + s_{34}^{k-1}) + \mathbf{v_{34}}[2]s_{34}^k + \mathbf{u_{43}}[2](s_{43}^1 + s_{43}^2 + ... + s_{43}^{k-1}) + \mathbf{v_{43}}[2]s_{43}^k$
3	$\mathbf{u_{12}}[3](s_{12}^1 + ... + s_{12}^{k-2} + s_{12}^k) + \mathbf{v_{12}}[3]s_{12}^{k-1} + \mathbf{u_{21}}[3](s_{21}^1 + ... + s_{21}^{k-2} + s_{21}^k) + \mathbf{v_{21}}[3]s_{21}^{k-1}$
	$+\mathbf{u_{34}}[3](s_{34}^1 + ... + s_{34}^{k-2} + s_{34}^k) + \mathbf{v_{34}}[3]s_{34}^{k-1} + \mathbf{u_{43}}[3](s_{43}^1 + ... + s_{43}^{k-2} + s_{43}^k) + \mathbf{v_{43}}[3]s_{43}^{k-1}$
$\vdots$	$\vdots$
k	$\mathbf{u_{12}}[k](s_{12}^1 + s_{12}^3 + ... + s_{12}^k) + \mathbf{v_{12}}[k]s_{12}^2 + \mathbf{u_{21}}[k](s_{21}^1 + s_{21}^3 + ... + s_{21}^k) + \mathbf{v_{21}}[k]s_{21}^2$
	$+\mathbf{u_{34}}[k](s_{34}^1 + s_{34}^3 + ... + s_{34}^k) + \mathbf{v_{34}}[k]s_{34}^2 + \mathbf{u_{43}}[k](s_{43}^1 + s_{43}^3 + ... + s_{43}^k) + \mathbf{v_{43}}[k]s_{43}^2$
k+1	$\mathbf{u_{12}}[k+1](s_{12}^1 + s_{12}^2 + ... + s_{12}^k) + \mathbf{u_{21}}[k+1](s_{21}^1 + s_{21}^2 + ... + s_{21}^k)$
	$+\mathbf{u_{34}}[k+1](s_{34}^1 + s_{34}^2 + ... + s_{34}^k) + \mathbf{u_{43}}[k+1](s_{43}^1 + s_{43}^2 + ... + s_{43}^k)$

TABLE III

SIGNALS SENT BY RELAY IN $k+1$ TIME SLOTS.

Time Slot	Knowledge of Signals at the Relay			
1	$\sum_{m=1}^k s_{12}^m$,	$\sum_{m=1}^k s_{21}^m$,	$\sum_{m=1}^k s_{34}^m$,	$\sum_{m=1}^k s_{43}^m$
2	$\sum_{m=1}^{k-1} s_{12}^m$, s_{12}^k	$\sum_{m=1}^{k-1} s_{21}^m$, s_{21}^k	$\sum_{m=1}^{k-1} s_{34}^m$, s_{34}^k	$\sum_{m=1}^{k-1} s_{43}^m$, s_{43}^k
3	$\sum_{m=1}^{k-2} s_{12}^m$, s_{12}^k, s_{12}^{k-1},	$\sum_{m=1}^{k-2} s_{21}^m$, s_{21}^k, s_{21}^{k-1},	$\sum_{m=1}^{k-2} s_{34}^m$, s_{34}^k, s_{34}^{k-1},	$\sum_{m=1}^{k-2} s_{43}^m$, s_{43}^k, s_{43}^{k-1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
k	$s_{12}^1, ..., s_{12}^k$,	$s_{21}^1, ..., s_{21}^k$,	$s_{34}^1, ..., s_{34}^k$,	$s_{43}^1, ..., s_{43}^k$
k+1	$s_{12}^1, ..., s_{12}^k$,	$s_{21}^1, ..., s_{21}^k$,	$s_{34}^1, ..., s_{34}^k$,	$s_{43}^1, ..., s_{43}^k$

TABLE II

SIGNALS DECODED AT THE RELAY IN $k+1$ TIME SLOTS.

Consider now receiver 1 (other receivers follow analogously). To prevent undesired signals from reaching receiver 1, we design the following beamforming vectors as:

$$\mathbf{u_{34}}[l], \mathbf{v_{34}}[l] \in null(\mathbf{h}_{\mathbf{R1}}^*[l]), \quad l \in \{2, 3, ..., k+1\}. \tag{34}$$

To align the k interference signals as $\sum_{m=1}^k s_{43}^m$ at receiver 1, we design the beamforming vectors to satisfy:

$$\mathbf{h}_{\mathbf{R1}}^*[l]\mathbf{v_{43}}[l] - \mathbf{h}_{\mathbf{R1}}^*[l]\mathbf{u_{43}}[l] = 2h_{41}[l], \quad l \in \{2, 3, ..., k\}. \tag{35}$$

Note that we can always construct such beamforming vectors $\mathbf{u_{ij}}[l], \mathbf{v_{ij}}[l]$ as in (27) and (28) in Section III (e.g. for $\mathbf{u_{34}}[l]$ and $\mathbf{v_{34}}[l]$, simply replace [2] by $[l]$ in (27) and (28)).

To decode the k desired signals $s_{21}^1, s_{21}^2, ..., s_{21}^k$ at receiver 1 (similarly for the other 3 receivers), the $k \times k$ matrix in Fig. 2 must be full-rank. To decode the sum of k interference signals $\sum_{m=1}^k s_{43}^m$ at receiver 1, the 2×2 matrix in (36) must be full-rank. Since the beamforming vectors $\mathbf{u_{ij}}[l], l \in \{2, 3, ..., k+1\}$ and $\mathbf{v_{ij}}[l], l \in \{2, 3, ..., k\}$ were designed independently of $\mathbf{h}_{\mathbf{Rj}}^*[l], l \in \{2, 3, ..., k+1\}$, and the channel coefficients are all drawn from a continuous random distribution, both matrices in Fig. 2 and (36) have full rank almost surely.

$$\begin{bmatrix} h_{21}[1] & h_{41}[1] \\ \mathbf{h}_{\mathbf{R1}}^*[k+1]\mathbf{u_{21}}[k+1] & \mathbf{h}_{\mathbf{R1}}^*[k+1]\mathbf{u_{43}}[k+1] \end{bmatrix} \tag{36}$$

Decoding: Now each receiver is able to decode k desired signals and a sum of k interference signals. For example, at receiver 1, we have $k+1$ equations:

$$Y_1[1] = h_{21}[1]\sum_{m=1}^{k} s_{21}^m + h_{41}[1]\sum_{m=1}^{k} s_{43}^m$$

$$Y_1[2] = (h_{21}[2] + \mathbf{h}_{\mathbf{R1}}^*[2]\mathbf{u_{21}}[2])\sum_{m=1}^{k-1} s_{21}^m + (-h_{21}[2]$$

$$+ \mathbf{h}_{\mathbf{R1}}^*[2]\mathbf{v_{21}}[2])s_{21}^k + (h_{41}[2] + \mathbf{h}_{\mathbf{R1}}^*[2]\mathbf{u_{43}}[2])\sum_{m=1}^{k} s_{43}^m + \text{SI}$$

$$Y_1[3] = (h_{21}[3] + \mathbf{h}_{\mathbf{R1}}^*[3]\mathbf{u_{21}}[3])\sum_{m=1, m\neq k-1}^{k} s_{21}^m + (-h_{21}[3]$$

$$+ \mathbf{h}_{\mathbf{R1}}^*[3]\mathbf{v_{21}}[3])s_{21}^{k-1} + (h_{41}[3] + \mathbf{h}_{\mathbf{R1}}^*[3]\mathbf{u_{43}}[3])\sum_{m=1}^{k} s_{43}^m + \text{SI}$$

...

$$\begin{bmatrix} h_{21}[1] & h_{21}[1] & \dots & h_{21}[1] & h_{21}[1] \\ h_{21}[2]+\mathbf{h}_{\mathbf{R1}}^*[2]\mathbf{u_{21}}[2] & h_{21}[2]+\mathbf{h}_{\mathbf{R1}}^*[2]\mathbf{u_{21}}[2] & \dots & h_{21}[2]+\mathbf{h}_{\mathbf{R1}}^*[2]\mathbf{u_{21}}[2] & -h_{21}[2]+\mathbf{h}_{\mathbf{R1}}^*[2]\mathbf{v_{21}}[2] \\ h_{21}[3]+\mathbf{h}_{\mathbf{R1}}^*[3]\mathbf{u_{21}}[3] & h_{21}[3]+\mathbf{h}_{\mathbf{R1}}^*[3]\mathbf{u_{21}}[3] & \dots & -h_{21}[3]+\mathbf{h}_{\mathbf{R1}}^*[3]\mathbf{v_{21}}[3] & h_{21}[3]+\mathbf{h}_{\mathbf{R1}}^*[3]\mathbf{u_{21}}[3] \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ h_{21}[k]+\mathbf{h}_{\mathbf{R1}}^*[k]\mathbf{u_{21}}[k] & -h_{21}[k]+\mathbf{h}_{\mathbf{R1}}^*[k]\mathbf{v_{21}}[k] & \dots & h_{21}[k]+\mathbf{h}_{\mathbf{R1}}^*[k]\mathbf{u_{21}}[k] & h_{21}[k]+\mathbf{h}_{\mathbf{R1}}^*[k]\mathbf{u_{21}}[k] \end{bmatrix}$$

Fig. 2. Channel coefficient matrix for k desired signals at receiver 1.

$$Y_1[k] = (h_{21}[k] + \mathbf{h}_{\mathbf{R1}}^*[k]\mathbf{u_{21}}[k]) \sum_{m=1, m\neq 2}^{k} s_{21}^m + (-h_{21}[k] + \mathbf{h}_{\mathbf{R1}}^*[k]\mathbf{v_{21}}[k]) s_{21}^2 + (h_{41}[k] + \mathbf{h}_{\mathbf{R1}}^*[k]\mathbf{u_{43}}[k]) \sum_{m=1}^{k} s_{43}^m + \text{SI}$$

$$Y_1[k+1] = \mathbf{h}_{\mathbf{R1}}^*[k+1]\mathbf{u_{21}}[k+1] \sum_{m=1}^{k} s_{21}^m + \mathbf{h}_{\mathbf{R1}}^*[k+1]\mathbf{u_{43}}[k+1] \sum_{m=1}^{k} s_{43}^m + \text{SI}$$

Note all self-interference (SI) signals can be subtracted at the receiver(s). The first and the last equation are used to decode the sum of k interference signals $\sum_{m=1}^{k} s_{43}^m$ which are then subtracted from the first k equations. Finally, the k desired signals can be decoded from the first k interference-free equations. A similar decoding procedure follows for the other 3 receivers. Therefore, $4k$ desired messages are obtained in $k+1$ time slots, i.e. $\frac{4k}{k+1}$ DoF is achievable, which tends to 4 DoF when $k \to \infty$. □

Remark 2: We have showed in [3] that the DoF of the two-way interference channel is 2; Theorem 1 implies that the addition of a non causal 4-antenna relay can increase the DoF of the two-way IC to 4 – it essentially cancels out all interference in both directions simultaneously. This may have interesting design implications for two-way interference networks – i.e. the addition of a relay (perhaps a pico-cell) could double, in a DoF sense, the rates achievable.

Remark 3: To achieve 4 DoF we have assumed full duplex operation (except for the 1st and last slots). If instead all nodes operate in half-duplex mode, intuitively the DoF will halved, i.e. 2. Indeed, it is trivial to achieve 2 DoF in half-duplex setup: In the first time slot, all 4 users transmit a message, and the 4-antenna relay listens and decodes all 4 messages using a zero-forcing decoder. At time slot 2, the relay broadcasts signals and all users listen. By careful choice of beamforming vectors, each receiver receives only their desired message in this time slot. Therefore 4 desired messages are obtained in 2 time slots, i.e. 2 DoF is achievable.

V. Conclusion

We demonstrated that the degrees of freedom of the full-duplex two-way interference channel with a non causal 4-antenna relay is equal to 4, which indicates relay is able to cancel out *all* the interference in both directions simultaneously. Consider a more practical *causal* relay assisted full-duplex two-way IC and provide numerical simulations (omitted here due to space limitation) to validate our theoretical results can be future work.

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