Nearly Optimal Non-Gaussian Codes for the Gaussian Interference Channel

(Invited Paper)

Alex Dytso, Daniela Tuninetti and Natasha Devroye
University of Illinois at Chicago, Chicago IL 60607, USA,
Email: odytso2, danielat, devroye @ uic.edu

***Abstract*—Recent work demonstrated that for the two-user Gaussian Interference Channel (G-IC) sub-optimal point-to-point codes can outperform optimal (Gaussian) point-to-point codes. However, it is not clear how far from capacity such sub-optimal codes operate. This work demonstrates a family of sub-optimal codes, generated from a mixture of Gaussian and discrete random variables, that is optimal up to an additive gap for the G-IC.**

The developed tools are of interest on their own and can be used in a variety of channel models. For example, it can be shown that the capacity of the block-asynchronous G-IC where the decoders are prevented from decoding the interfering signals is to within an additive gap of the capacity of the classical G-IC where the receivers are fully synchronized and informed about the interfering codebooks.

I. INTRODUCTION

The memoryless real-valued additive white Gaussian noise interference channel (G-IC) has input-output relationship

$$Y_1^n = h_{11}X_1^n + h_{12}X_2^n + Z_1^n, \tag{1a}$$
$$Y_2^n = h_{21}X_1^n + h_{22}X_2^n + Z_2^n, \tag{1b}$$

where $X_j^n := (X_{j1}, \cdots X_{jn})$ and $Y_j^n := (Y_{j1}, \cdots Y_{jn})$ are the length-n vector inputs and outputs, respectively, for user $j \in [1:2]$, the noise vectors Z_j^n have i.i.d. zero-mean unit-variance Gaussian components, for n the block length. The input X_j^n is a function of the independent message W_j that is uniformly distributed on $[1:2^{nR_j}]$, where R_j is the rate for user $j \in [1:2]$, and is subject to a per-block power constraint $\frac{1}{n}\sum_{i=1}^n X_{ji}^2 \le 1$. Receiver $j \in [1:2]$ wishes to recover W_j from the channel output Y_j^n with arbitrarily small probability of error. Achievable rates and capacity region are defined in the usual way [1].

For sake of simplicity, we shall focus from now on on the *symmetric* G-IC only, defined as $|h_{11}|^2 = h_{22}|^2 = \mathsf{S} \ge 0$, $|h_{12}|^2 = |h_{21}|^2 = \mathsf{I} \ge 0$ and denote the capacity region as $\mathcal{C}(\mathsf{S}, \mathsf{I})$. The restriction to the symmetric case is just to reduce the number of parameters in our derivations; for more general asymmetric settings the reader may refer to [2].

Past Work. The general information stable two-user interference channel was first introduced in [3] and its capacity is characterized as

$$\mathcal{C} = \lim_{n\to\infty} \mathsf{co} \bigcup_{P_{X_1^n X_2^n} = P_{X_1^n} P_{X_2^n}} \left\{ \begin{array}{l} R_1 \le \frac{1}{n} I(X_1^n; Y_1^n) \\ R_2 \le \frac{1}{n} I(X_2^n; Y_2^n) \end{array} \right\} \tag{2}$$

where $\mathsf{co}(\cdot)$ denotes the convex hull operation. Unfortunately, the capacity expression in (2) is considered "uncomputable" in a sense that it is not known explicitly how to characterize the input distributions that attain its convex closure. Moreover, it is not clear whether there exists an equivalent single-letter form for (2) in general. For the G-IC an equivalent single-letter form is known in the strong interference region [4], given by $\mathsf{S} \le \mathsf{I}$ in the symmetric case.

Because of it "uncomputability," the capacity expression in (2) has received little attention, except for [5] where it was shown that jointly Gaussian inputs may not be optimal. Instead, the field has focussed on finding alternative ways to characterize single-letter inner and outer bounds. The best known inner bound is the Han and Kobayashi (HK) achievable scheme [6], which is capacity achieving in all cases where the whole capacity region is known [1].

Except for the strong interference regime, the sum-capacity of the G-IC is known exactly only for the Z-channel and in the noisy interference regime, the latter defined by $\sqrt{\frac{\mathsf{I}}{\mathsf{S}}}(1+\mathsf{I}) \le \frac{1}{2}$ in the symmetric case, for which i.i.d. Gaussian inputs in (2) are optimal [1]. So, instead of pursuing exact results, the community has focussed on giving performance guarantees on approximations of the capacity region. In [7] the authors showed that the HK scheme with i.i.d. Gaussian inputs and without time-sharing is optimal to within 1/2 bit, irrespective of the channel parameters, of the outer bound given by:

Proposition 1. *The capacity region of the symmetric G-IC is contained in*

$$\mathcal{R}_{out} = \Big\{ \quad R_1, R_2 \le \mathsf{I_g}(\mathsf{S}), \ \textit{cut-set bound}, \tag{3a}$$

$$R_1 + R_2 \le \Big[\mathsf{I_g}(\mathsf{S}) - \mathsf{I_g}(\mathsf{I})\Big]^+ + \mathsf{I_g}(\mathsf{I}+\mathsf{S}), \ \textit{[8]}, \tag{3b}$$

$$R_1 + R_2 \le 2\mathsf{I_g}\left(\mathsf{I} + \frac{\mathsf{S}}{1+\mathsf{I}}\right), \ \textit{[7]}, \tag{3c}$$

$$R_1 + 2R_2, 2R_1 + R_2 \le \mathsf{I_g}(\mathsf{S}+\mathsf{I}) + \mathsf{I_g}\left(\mathsf{I} + \frac{\mathsf{S}}{1+\mathsf{I}}\right) + \Big[\mathsf{I_g}(\mathsf{S}) - \mathsf{I_g}(\mathsf{I})\Big]^+, \ \textit{[7]}, \tag{3d}$$

For the classical G-IC this outer bound is tight in strong interference $\{\mathsf{I} \ge \mathsf{S}\}$ *[9] and achievable to within* $1/2$ *bit otherwise [7].*

Contribution. In this work we focus on the following capacity inner bound obtained by using i.i.d. inputs in (2)

$$\mathcal{R}_{\text{in}}^{\text{TIN}}(\mathsf{S},\mathsf{I}) = \bigcup_{P_{X_1X_2}=P_{X_1}P_{X_2}} \left\{ \begin{array}{l} R_1 \le I(X_1;Y_1) \\ R_2 \le I(X_2;Y_2) \end{array} \right\}, \quad (4)$$

commonly referred to as the "treating interference as noise" (TIN) inner bound. Note that the TIN region may not be convex because a time-sharing/convex hul operation is not considered. Our major contribution is to show that *i.i.d. mixed inputs* (i.e., a superposition of discrete and Gaussian random variables) in the TIN region in (4) achieve the capacity region outer bound in Prop. 1 to within a gap.

Notation convention. Throughout the paper we use the following notation. Lower case variables are instances of upper case random variables (r.v.) which take on values in calligraphic alphabets. We let

$$\mathsf{N}_\mathsf{d}(x) := \left\lfloor \sqrt{1+x} \right\rfloor, \quad (5)$$

$$\mathsf{I}_\mathsf{g}(x) := \frac{1}{2}\log(1+x), \quad (6)$$

$$\mathsf{Gap}(x,y) := \frac{1}{2}\log\left(x\frac{\pi \mathrm{e}}{3}\right) + \frac{1}{2}\log\left(1 + y\frac{(1+1/2\log(1+\min(\mathsf{S},\mathsf{I})))^2}{\gamma^2}\right), \quad (7)$$

where the subscripts d and g remind the reader that discrete and Gaussian, respectively, inputs are involved. $H(X)$ is the entropy of the discrete r.v. X and $[x]^+ := \max\{0,x\}$. If A is a r.v. we denote its support by $\mathsf{supp}(A)$. The symbol $|\cdot|$ denotes: $|\mathcal{A}|$ is the cardinality of the set $\mathcal{A}$, $|X|$ is the cardinality of $\mathsf{supp}(X)$ of the random variable X, or $|x|$ is the absolute value of the real-valued x. $d_{\min(\mathcal{S})} := \min_{i\neq j: s_i,s_j\in\mathcal{S}} |s_i - s_j|$ is the minimum distance among the points in the set $\mathcal{S}$. With some abuse of notation we use $d_{\min(X)}$ to denote $d_{\min(\mathsf{supp}(X))}$ for a r.v. X. $X \sim \mathcal{N}(\mu,\sigma^2)$ denotes Gaussian r.v. with mean μ and variance σ^2. $X \sim \mathsf{PAM}\left(N, d_{\min(X)}\right)$ denotes the uniform probability mass function over a zero-mean Pulse Amplitude Modulation (PAM) constellation with $|\mathsf{supp}(X)| = N$ points, minimum distance $d_{\min(X)}$, and average energy $\mathbb{E}[X^2] = d^2_{\min(X)}\frac{N^2-1}{12}$. $m(\mathcal{S})$ denotes Lebesgue measure of the set $\mathcal{S}$. For $i \in [1:2]$ we let $i' \in [1:2]$ to be $i' \neq i$.

II. Main Tools

In the rest of the paper, due to space limitations, many of the proofs are omitted and may be found in [2]. At the core of our proofs is the lower bound on the rate achieved by a discrete input on a point-to-point additive noise channel.

Since, there are many lower bound available in the literature, it is worthwhile to compare these bounds, in order to see different trade offs between them.

Proposition 2. *Let X_D be a discrete random variable, whose support has size N, minimum distance $d_{\min(X_D)}$, and average power $\mathbb{E}[X_D^2]$. Let Z be a zero-mean unit-variance Gaussian r.v. independent of X_D. Then*

$$[H(X_D) - \mathsf{gap}] := \mathsf{I}_\mathsf{d}(X_D) \le I(X_D; X_D+Z) \quad (8a)$$
$$\le \min\left(H(X_D), \mathsf{I}_\mathsf{g}(\mathbb{E}[X_D^2])\right),$$

and where $\mathsf{gap} \le \min(\mathsf{gap}_{(8b)}, \mathsf{gap}_{(8c)}, \mathsf{gap}_{(8d)})$

$$\mathsf{gap}_{(8b)} := \xi\log\frac{1}{\xi} + (1-\xi)\log\frac{1}{1-\xi} + \xi\log(N-1),$$
$$\xi := 2Q\left(\frac{d_{\min(X_D)}}{2}\right), \ \textit{Ozarow-Wyner-A, [10]} \quad (8b)$$

$$\mathsf{gap}_{(8c)} := \frac{1}{2}\log\left(\frac{2\pi\mathrm{e}}{12}\right) + \frac{1}{2}\log\left(1 + \frac{12}{d^2_{\min(X_D)}}\right),$$
Ozarow-Wyner-B, [10] (8c)

$$\mathsf{gap}_{(8d)} := \frac{1}{2}\log\left(\frac{\mathrm{e}}{2}\right) + \log\left(1 + (N-1)\mathrm{e}^{-d^2_{\min(X_D)}/4}\right),$$
DTD-ITA'14-A, [11]. (8d)

The bound in (8d) is special case of the following bound:

Proposition 3. *For X_D and Z as defined in Prop. 2*

$$I(X_D; X_D+Z) \ge \left[-\log\left(\sum_{(i,j)\in[1:N]^2} p_i p_j \frac{1}{\sqrt{4\pi}}\mathrm{e}^{-\frac{(s_i-s_j)^2}{4}}\right) - \frac{1}{2}\log(2\pi\mathrm{e})\right]^+,$$
DTD-ITA'14-B, [11] (9)

where $p_i = \mathbb{P}[X_D = s_i]$.

Next, we numerically compare the lower bounds in Prop. 2 and Prop. 3 for the Gaussian noise channel with a PAM input, which is asymptotically capacity achieving at high SNR [10]. In Fig. 1 we plot the gap to capacity, i.e., the difference between the channel capacity, $\mathsf{I}_\mathsf{g}(\mathsf{S})$ and different lower bounds on $I(X_D; \sqrt{\mathsf{S}}\ X_D + Z)$. Here S represents the SNR at the receiver, $X_D \sim \mathsf{PAM}\left(N, \sqrt{\frac{12}{N^2-1}}\right)$ is the input with $N = \mathsf{N}_\mathsf{d}(\mathsf{S}) = \left\lfloor\sqrt{1+\mathsf{S}}\right\rfloor \approx \mathsf{S}^{\frac{1}{2}}$. We observe:

1) The blue curve is the Ozarow-Wyner-B bound in (8c). This bound is asymptotically (for $\mathsf{S} \ge 30$dB) to within 0.754 bits of capacity, 2) the magenta curve is the Ozarow-Wyner-A bound in (8b). This bound is to within $O(\log(\mathsf{S}))$ of capacity (i.e., straight line as a function of $\mathsf{S}|_{\text{dB}}$), 3) the cyan curve is the simple DTD-ITA'14-A bound in (8d). Here we used $N = \mathsf{N}_\mathsf{d}(\mathsf{S}^{1-\epsilon}) \approx \mathsf{S}^{\frac{1-\epsilon}{2}}$ with $\epsilon = \max\left(0, \frac{\log(\frac{1}{6}\ln(\mathsf{S}))}{\log(\mathsf{S})}\right)$. This choice of ϵ was derived in [11, Thm. 3] in order to have a $O(\log\log(\mathsf{S}))$ gap to capacity. Had we chosen $\epsilon = 0$ then we could only achieve a 'gap' of $O(\log(\mathsf{S}))$. Similarly, for the Ozarow-Wyner-A, had we choose the same $\epsilon = \max\left(0, \frac{\log(\frac{1}{6}\ln(\mathsf{S}))}{\log(\mathsf{S})}\right)$ a similar $O(\log\log(\mathsf{S}))$ gap would have been observed, 4) the green curve is the DTD-ITA'14-B bound in (9), which achieves asymptotically (for $\mathsf{S} \ge 30$dB) to within 0.36 bits of capacity. The quantity $\frac{1}{2}\log\left(\frac{\pi\mathrm{e}}{6}\right) \approx 0.254$ is also shown for reference in Fig. 1; this is the "shaping loss" for a one-dimensional infinite lattice. The "zig-zag" behavior of the curves at low SNR is due to the floor operation in choosing $N = \left\lfloor\sqrt{1+\mathsf{S}}\right\rfloor$.

We observe a qualitatively different behavior at high SNR: the Ozarow-Wyner-B bound in (8c) (blue curve) and the DTD-ITA'14-B bound in (9) (green curve) result in a constant gap, while the Ozarow-Wyner-A bound in (8b) (magenta curve) and

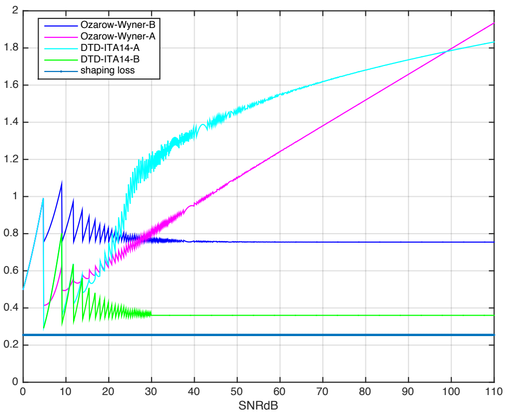


Fig. 1: Gap to capacity vs. S for different bounds for a PAM input on a Gaussian noise channel.

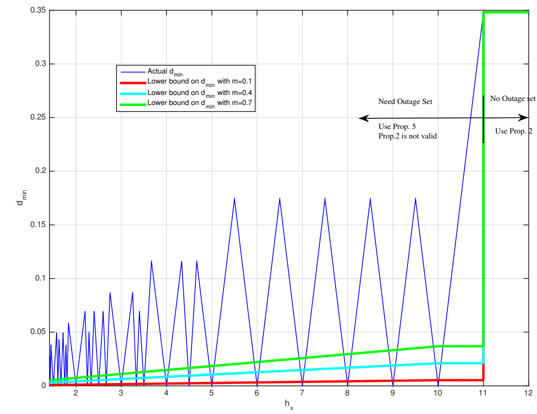


Fig. 2: Minimum distance (blue line) for the sum-set $h_x X + h_y Y$ as a function of h_x for fixed $h_y = 1$ and for $X \sim Y \sim \mathsf{PAM}\,(10, 1)$.

the DTD-ITA'14-A bound in (8d) (cyan curve) result in a gap that grows with SNR; this is in agreement with our intuition that for a constant gap in the latter two cases the number of points N must grow slower than $\mathsf{S}^{1/2}$. The smallest gap at high SNR for $N \approxeq \mathsf{S}^{1/2}$ is given by the DTD-ITA'14-B bound in (9) (green curve), however, this bound is unfortunately not amenable for closed form analytical evaluations, so in the following we shall use the Ozarow-Wyner-B bound in (8c) (blue curve) from Prop. 2 whose simplicity comes at the cost of a larger gap.

In the following we shall use a mixed input at each user of the G-IC; this implies that a receiver sees a linear combination of two discrete constellations; in order to apply Prop. 2 we need to lower bound the minimum distance of the resulting sum-set. The following set of sufficient conditions will play an important role in evaluating the minimum distance of mixed input constellations in our TIN inner bound.

Proposition 4. *Let* $(h_x, h_y) \in \mathbb{R}^2$ *be two constants. Let* $X \sim \mathsf{PAM}(|X|, d_{\min(X)})$ *and* $Y \sim \mathsf{PAM}(|Y|, d_{\min(Y)})$*. Then*

$$|h_x X + h_y Y| = |X||Y|,$$
$$d_{\min(h_x X + h_y Y)} = \min\left(|h_x| d_{\min(X)}, |h_y| d_{\min(Y)}\right),$$

under the following conditions

$$\text{either} \quad |Y||h_y| d_{\min(Y)} \le |h_x| d_{\min(X)} \quad (10a)$$
$$\text{or} \quad |X||h_x| d_{\min(X)} \le |h_y| d_{\min(Y)}. \quad (10b)$$

When Prop. 4 is not applicable we shall use:

Proposition 5. *Let* $X \sim \mathsf{PAM}(|X|, d_{\min(X)})$ *and* $Y \sim \mathsf{PAM}(|Y|, d_{\min(Y)})$*. Then for* $(h_x, h_y) \in \mathbb{R}^2$ *we have*

$$|h_x X + h_y X| = |X||Y| \text{ almost everywhere (a.e.)} \quad (11a)$$

and for any $\gamma > 0$ *there exists a set* $E \subseteq \mathbb{R}$ *such that for all* $(h_x, h_y) \in E$ *we have*

$$d_{\min(h_x X + h_y Y)} \ge \kappa_{\gamma,|X|,|Y|} \min\Big(|h_x| d_{\min(X)},$$
$$|h_y| d_{\min(Y)}, \max\left(\frac{|h_x| d_{\min(X)}}{|Y|}, \frac{|h_y| d_{\min(Y)}}{|X|}\right)\Big), \quad (11b)$$

where $\kappa_{\gamma,|X|,|Y|} = \frac{\gamma}{2(1+\ln(\max(|X|,|Y|)))}$ *and the complement of the set* E*, referred to as the "outage set," has* $m(E^c) \le \gamma$*.*

Fig. 2 shows an example of the behavior of the minimum distance. There are two distinct regions: one where minimum distance is a constant (for $h_x \ge 11$ and one where the minimum distance is a very irregular function. In Fig. 2 on the right of the vertical green line Prop. 4 is valid, this is the smooth region. On the left of the vertical green line, the irregular region, Prop. 5 must be used; in latter case, the minimum distance lower bound in (11b) holds for set of h_x's for which the blue line is above the red / cyan / green line, where the red, cyan and green lines represent a different value for the measure of the outage set.

III. Main Result

For the G-IC in (1) we now evaluate the TIN region in (4) with mixed inputs

$$X_i = \sqrt{1-\delta_i}\ X_{iD} + \sqrt{\delta_i}\ X_{iG},\ i \in [1:2]: \quad (12a)$$
$$X_{iD} \sim \mathsf{PAM}\left(N_i, \sqrt{\frac{12}{N_i^2 - 1}}\right), \quad (12b)$$
$$X_{iG} \sim \mathcal{N}(0,1), \quad (12c)$$
$$\mathbf{P} := [N_1, N_2, \delta_1, \delta_2] \in \mathbb{N} \times \mathbb{N} \times [0,1] \times [0,1], \quad (12d)$$

where the r.v X_{ij} are independent for $i \in [1:2],\ j \in \{D, G\}$. A careful choice of the parameters in $\mathbf{P}$ will lead to the desired results in different parameter regimes. From the TIN region in (4) we have:

Proposition 6. *For the G-IC the following region is achievable*

$$\mathcal{R}_{in}(\mathsf{S}, \mathsf{I}) := \bigcup_{\mathbf{P} \in \mathbb{N}^2 \times [0,1]^2} \mathcal{R}_{in}(\mathsf{S}, \mathsf{I}; \mathbf{P}), \quad (13a)$$

where $\mathcal{R}_{in}(\mathsf{S}, \mathsf{I}; \mathbf{P})$ *is a lower bound on the TIN region in* (4) *evaluated for the mixed input in* (12) *with fixed parameter vector* $\mathbf{P} := [N_1, N_2, \delta_1, \delta_2]$*, given by*

$$\mathcal{R}_{in}(\mathsf{S}, \mathsf{I}; \mathbf{P}) := \Big\{(R_1, R_2) : R_i \le \mathsf{I_d}\left(S_i\right) + \mathsf{I_g}\left(\frac{\mathsf{S}\delta_i}{1 + \mathsf{I}\delta_{i'}}\right)$$

$$- \min\left(\log(N_{i'}), \mathsf{I}_{\mathsf{g}}\left(\frac{\mathsf{I}(1-\delta_{i'})}{1+\mathsf{I}\delta_{i'}}\right)\right), i \in [1:2]\Bigg\}, \quad \text{(13b)}$$

and where the equivalent discrete constellations seen at the receivers are

$$S_i := \frac{\sqrt{1-\delta_i}\sqrt{\mathsf{S}}X_{iD} + \sqrt{1-\delta_{i'}}\sqrt{\mathsf{I}}X_{i'D}}{\sqrt{1+\mathsf{S}\delta_i+\mathsf{I}\delta_{i'}}},$$

for $i' \neq i \in [1:2]$ *and the functionals* $\mathsf{I}_{\mathsf{d}}(\cdot)$ *and* $\mathsf{I}_{\mathsf{g}}(\cdot)$ *are defined in* (8a) *and* (7)*, respectively.*

Theorem 7. *For the symmetric G-IC, the TIN achievable region in* (13a)*, and the outer bound in* (3) *are to within a gap of:*

- *Very Weak Interference:* $\mathsf{S} \geq \mathsf{I}(1+\mathsf{I})$*:* $\mathsf{gap} \leq \frac{1}{2}$ *bits,*
- *Moderately Weak Interference Type2:* $\mathsf{S} < \mathsf{I}(1+\mathsf{I})$, $\frac{1+\mathsf{S}}{1+\mathsf{I}+\frac{\mathsf{S}}{1+\mathsf{I}}} > \frac{1+\mathsf{I}+\frac{\mathsf{S}}{1+\mathsf{I}}}{1+\frac{\mathsf{S}}{1+\mathsf{I}}}$*:* $\mathsf{gap} \leq \mathsf{Gap}\left(\frac{608}{9}, 0\right) \approx 3.79$ *bits,*
- *Moderately Weak Interference Type1:* $\mathsf{I} \leq \mathsf{S}$, $\frac{1+\mathsf{S}}{1+\mathsf{I}+\frac{\mathsf{S}}{1+\mathsf{I}}} \leq \frac{1+\mathsf{I}+\frac{\mathsf{S}}{1+\mathsf{I}}}{1+\frac{\mathsf{S}}{1+\mathsf{I}}}$*:* $\mathsf{gap} \leq \mathsf{Gap}(16, 45)$ *bits, except for a set of measure* γ *for any* $\gamma \in (0, 1]$*,*
- *Strong Interference:* $\mathsf{S} < \mathsf{I} < \mathsf{S}(1+\mathsf{S})$*:* $\mathsf{gap} \leq \mathsf{Gap}(2, 8)$ *bits, except for a set of measure* γ *for any* $\gamma \in (0, 1]$*,*
- *Very Strong Interference:* $\mathsf{I} \geq \mathsf{S}(1+\mathsf{S})$*:* $\mathsf{gap} \leq \mathsf{Gap}(2, 0) \approx 1.25$ *bits.*

For the optimal choice of parameters $\mathbf{P}$ see [2, Table 1].

IV. Actual vs. Analytic Gap

Here we compare the gap derived in Thm. 7 to the actual gap evaluated numerically. The point is to show that our analytical closed-form (worst case scenario) bounds are quite conservative and thus underestimate the actual achievable rates. In Fig. 3 observe that

- the red line is the theoretical result from Thm. 7;
- the green line is the gap by lower bounding rates in Prop. 6 with the Ozarow-Wyner-B bound in Prop. 2, where the minimum distance of the received constellation was computed exactly (rather than lower bounded by Prop. 4 or Prop. 5);
- the magenta line is the gap by lower bounding rates in Prop. 6 with DTD-ITA'14-B in Prop. 3;
- the cyan line is the gap evaluated by numerical integration.

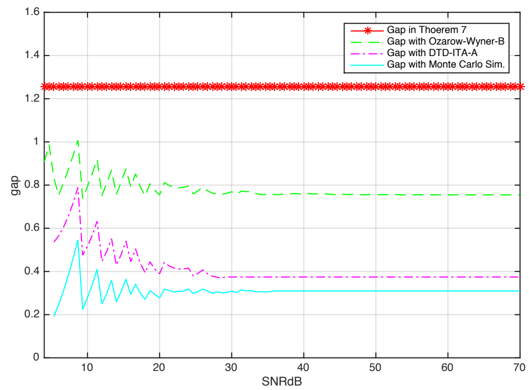
For example, in very strong interference regime, while the analytic gap (red line) is given by $\frac{1}{2}\log\left(\frac{2\pi e}{3}\right) \approx 1.25$ bits, the gap by lower bounding rates in Prop. 6 with the Ozarow-Wyner-B bound in Prop. 2 (green line), where the minimum distance of the received constellation was computed exactly (rather than lower bounded by Prop. 4) is approximately 0.75 bits. Moreover, the gap by lower bounding rates in Prop. 6 by the DTD-ITA'14-B bound' in Prop. 3 (magenta line), is approximately 0.37 bits; 4) the cyan line is the gap when symmetric rate is evaluated by numerical integation; the gap in this case tends to the ultimate "shaping loss" $\frac{1}{2}\log\left(\frac{\pi e}{6}\right) = 0.25$ bits at large S; this shows that the actual gap is about 1 bit lower than the theoretical gap for the same choice of parameters as in Thm. 7.

Fig. 3 also shows that the lower bound in Prop. 3 actually gives the tightest lower bound for the mutual information, but it is unfortunately not easy to deal with analytically.

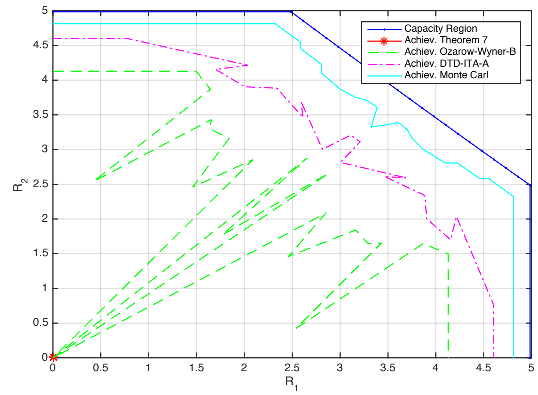
We next consider the symmetric G-IG in strong interference. Thm. 7 upper bounds the gap in this regime by $\mathsf{Gap}(4, 8)$ where $\gamma \in (0, 1]$ is the measure of the outage set (i.e., those channel gains for which the gap lower bound is not valid). If we were to make the measure of the outage set very small, then we could end up finding that the gap is actually larger than point-to-point capacity. Consider the case $\mathsf{S} = 30$ dB and $\mathsf{I} = \mathsf{S}^{1.49} = 44.7$ dB; with $\gamma = 0.1$ it easy to see that $\mathsf{Gap}(4, 8) \approx 6.977$ bits which is larger than the interference-free capacity $\mathsf{I}_{\mathsf{g}}(\mathsf{S}) = 4.9836$ bits. This implies that our bounding steps, done for the sake of analytical tractability and especially meaningful at high SNR, are too crude for this specific example (where our result states the trivial fact that zero rate for each user is achievable to within $\mathsf{I}_{\mathsf{g}}(\mathsf{S})$ bits). We aim to convey next that, despite the fact that the closed-form gap result underestimates the achievable rates, it nonetheless provides valuable insights into the performance of practical systems, that is, that TIN with discrete inputs performs quite well in the strong interference regime (where capacity is achieved by Gaussian codebooks and joint decoding of interfering and intended messages). To this end, Fig. 3b shows the achievable rate region for the symmetric G-IC with $\mathsf{S} = 30$ dB and $\mathsf{I} = \mathsf{S}^{1.49} = 44.7$ dB. We observe that while the analytic rate is zero the actual achievable rates can be quite high. The reason why the green region has so many 'ups and downs' is because the Ozarow-Wyner-B bound depends on the constellation through its minimum distance; as we already saw in Fig. 2, the minimum distance is very sensitive to the fractional values of the channel gains, which makes the corresponding bound looks very irregular. On the other hand, the magenta region is based on the lower bound in Prop. 3, which depends on the whole distance spectrum of the received constellation and as a consequence the corresponding bound looks smoother. The cyan region is the smoothest of all; its largest gap occurs at the symmetric rate point and is less than 0.7 bits – as opposed to the theoretical gap of 4.9836 bits. We thus conclude that, despite the large theoretical gap, a PAM input is quit competitive in this example. For completeness, in Fig. 3c and Fig. 3d plot the achievable regions in Weak Interference of Type 1 and Type 2 where similar conclusion can be made.

V. Conclusion

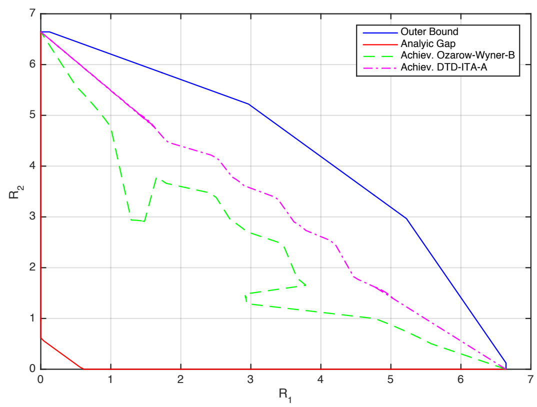
In this paper we proved that a very simple, generally applicable, lower bound that neither requires joint decoding nor time sharing and is optimal to within an additive gap (either constant uniformly over the channel gains, or of order $O(\log\log(\mathsf{S}))$ up to an outage set of controllable measure). Our result demonstrates that properly accounting for the distribution of the interference (i.e., not Gaussian with our mixed inputs) when treating interference as noise results in



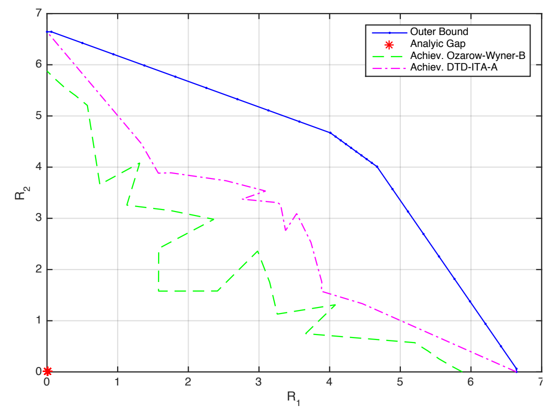
(a) Gap in the very strong interference vs. S: $\mathsf{I} = (\mathsf{S}\ (1+\mathsf{S}))^{1.2} \approx \mathsf{S}^{2.4}$.



(b) Rate region in strong interference: $\mathsf{S} = 30$ dB and $\mathsf{I} = \mathsf{S}^{1.49} = 44.7$ dB.



(c) Rate region in weak 2 interference: $\mathsf{S} = 40$ dB and $\mathsf{I} = \mathsf{S}^{0.6} = 24$ dB.



(d) Rate region in weak 1 interference: $\mathsf{S} = 40$ dB and $\mathsf{I} = \mathsf{S}^{0.7} = 28$ dB and $\gamma = 0.2$.

Fig. 3: Comparing analytic and numerical results. Inputs for analytical and numerical results are those that are used in Thm. 7.

near optimal rates for all channel parameters. Interestingly the result also holds for G-IC with no block synchronization and no codebook knowledge. This is due to the fact that TIN does not require synchronization or codebook knowledge, for in detail discussion see, [2] and [12].

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