i.i.d. Mixed Inputs and Treating Interference as Noise are gDoF Optimal for the Symmetric Gaussian Two-user Interference Channel

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***Abstract*—While a multi-letter limiting expression of the capacity region of the two-user Gaussian interference channel is known, capacity is generally considered to be open as this is not computable. Other computable capacity outer bounds are known to be achievable to within 1/2 bit using Gaussian inputs and joint decoding in the simplified Han and Kobayashi (single-letter) achievable rate region. This work shows that the simple scheme known as "treating interference as noise" without time-sharing attains the capacity region outer bound of the symmetric Gaussian interference channel to within either a constant gap, or a gap of order $O(\log\log(\mathrm{SNR}))$, for all parameter regimes. The scheme is therefore optimal in the generalized Degrees of Freedom (gDoF) region sense almost surely. The achievability is obtained by using *i.i.d. mixed inputs* (i.e., a superposition of discrete and Gaussian random variables) in the multi-letter capacity expression, where the optimal number of points in the discrete part of the inputs, as well as the optimal power split among the discrete and continuous parts of the inputs, are characterized in closed form. An important practical implication of this result is that the discrete part of the inputs behaves as a "common message" whose contribution can be removed from the channel output, even though joint decoding is not employed. Moreover, time-sharing may be mimicked by varying the number of points in the discrete part of the inputs.**

I. Introduction

The memoryless real-valued additive white Gaussian noise interference channel (G-IC) has input-output relationship

$$Y_1^n = h_{11}X_1^n + h_{12}X_2^n + Z_1^n, \quad \text{(1a)}$$
$$Y_2^n = h_{21}X_1^n + h_{22}X_2^n + Z_2^n, \quad \text{(1b)}$$

where $X_j^n := (X_{j1}, \cdots X_{jn})$ and $Y_j^n := (Y_{j1}, \cdots Y_{jn})$ are the length-n vector inputs and outputs, respectively, for user $j \in [1:2]$, the noise vectors Z_j^n have i.i.d. zero-mean unit-variance Gaussian components, for n the block length. The input X_j^n is a function of the independent message W_j that is uniformly distributed on $[1:2^{nR_j}]$, where R_j is the rate for user $j \in [1:2]$, and is subject to a per-block power constraint $\frac{1}{n}\sum_{i=1}^n X_{ji}^2 \le 1$. Receiver $j \in [1:2]$ wishes to recover W_j from the channel output Y_j^n with arbitrarily small probability of error. Achievable rates and capacity region are defined in the usual way [1].

For sake of simplicity, we shall focus from now on on the *symmetric* G-IC only, defined as

$$|h_{11}|^2 = |h_{22}|^2 = \mathsf{S} \ge 0,\ |h_{12}|^2 = |h_{21}|^2 = \mathsf{I} \ge 0$$

and denote the capacity region as $\mathcal{C}(\mathsf{S},\mathsf{I})$. The restriction to the symmetric case is just to reduce the number of parameters in our derivations; we believe that the results of this paper hold for the general asymmetric case.

Past Work. The general information stable two-user interference channel was first introduced in [2] and its capacity may be characterized as

$$\mathcal{C} = \lim_{n\to\infty} \mathsf{co} \bigcup_{P_{X_1^n X_2^n} = P_{X_1^n} P_{X_2^n}} \left\{ \begin{array}{l} R_1 \le \frac{1}{n} I(X_1^n; Y_1^n) \\ R_2 \le \frac{1}{n} I(X_2^n; Y_2^n) \end{array} \right\} \quad (2)$$

where $\mathsf{co}(\cdot)$ denotes the convex hull operation. Unfortunately, the capacity expression in (2) is considered "uncomputable" in a sense that it is not known explicitly how to characterize the input distributions that attain its convex closure. Moreover, it is not clear whether there exists an equivalent single-letter form for (2) in general. For the G-IC an equivalent single-letter form is known in the strong interference region [3] given by $\mathsf{S} \le \mathsf{I}$ in the symmetric case.

Because of it "uncomputability," the capacity expression in (2) has received little attention, except for [4] where it was shown that jointly Gaussian inputs may not be optimal. Instead, the field has focussed on finding alternative ways to characterize single-letter inner and outer bounds. The best known inner bound is the Han and Kobayashi (HK) achievable scheme [5], which is capacity achieving in all cases where capacity is known [1].

Except for the strong interference regime, the sum-capacity of the G-IC is known exactly only for the Z-channel and in the noisy interference regime, defined by $\sqrt{\frac{\mathsf{I}}{\mathsf{S}}}(1+\mathsf{I}) \le \frac{1}{2}$ in the symmetric case, for which i.i.d. Gaussian inputs in (2) are optimal [1]. So, instead of pursuing exact results, the community has recently focussed on giving performance guarantees on approximations of the capacity region. In [6] the authors showed that the HK scheme with Gaussian inputs and without time-sharing is optimal to within 1/2 bit/sec/Hz, irrespective of the channel parameters. This constant gap result provides an exact characterization of the generalized Degrees of Freedom (gDoF) region defined as

$$\mathcal{D}(\alpha) := \Big\{ (d_1, d_2) : d_i := \lim_{\mathsf{S}\to\infty} \frac{R_i(\mathsf{S}, \mathsf{I} = \mathsf{S}^\alpha)}{\frac{1}{2}\log(1+\mathsf{S})}, i \in [1:2],$$
$$(R_1, R_2) \text{ is achievable} \Big\}, \quad (3)$$

and where

$$\mathcal{D}(\alpha): d_1 \le 1,\ d_2 \le 1 \tag{4a}$$
$$d_1 + d_2 \le [1-\alpha]^+ + \max(\alpha, 1) \tag{4b}$$
$$d_1 + d_2 \le \max(1-\alpha, \alpha) + \max(1-\alpha, \alpha) \tag{4c}$$
$$2d_1 + d_2 \le [1-\alpha]^+ + \max(1, \alpha) + \max(1-\alpha, \alpha) \tag{4d}$$
$$d_1 + 2d_2 \le [1-\alpha]^+ + \max(1, \alpha) + \max(1-\alpha, \alpha). \tag{4e}$$

Contribution. In this work we focus on the following capacity inner bound obtained by using i.i.d. inputs in (2)

$$\mathcal{R}_{\text{in}}^{\text{TIN}}(\mathsf{S}, \mathsf{I}) = \bigcup_{P_{X_1 X_2} = P_{X_1} P_{X_2}} \left\{ \begin{array}{l} R_1 \le I(X_1; Y_1) \\ R_2 \le I(X_2; Y_2) \end{array} \right\}, \tag{5}$$

commonly referred to as the "treating interference as noise" (TIN) inner bound. Note that the TIN region may not be convex because a time-sharing / convex-hull operation is not considered. Our major contribution is to show that *i.i.d. mixed inputs* (i.e., a superposition of discrete and Gaussian random variables) in the TIN region in (5) achieve the capacity region outer bound to within a gap and thus are optimal for the whole gDoF region in (4). This work extends our past work in [7] where we demonstrated that i.i.d. mixed inputs achieve the symmetric sum-capacity (while here we consider the whole capacity region) to within a constant gap or to within $O(\log\log \mathsf{S})$ up to a outage set whose Lebesgue measure is a controllable parameter.

We note that a similar result (i.e., optimality of TIN in all parameter regimes) was pointed out in [1, Remark 6.12]. However, the proof is not constructive and is based on showing that TIN is optimal for the linear deterministic model (LDA), and then using a universal gap between the LDA and G-IC to arrive at the result. Unfortunately, this approach results in a very large gap and characterization of the inputs for the G-IC from the LDA is not immediate. On the other hand, our proof is constructive and provides, in closed form, the optimal number of points in the discrete part of the inputs, as well as the optimal power split among the discrete and continuous parts of the inputs. An important practical implication of this result is that the discrete part of the inputs behaves as a "common message" whose contribution can be removed from the channel output, even though joint decoding is not employed. Moreover, time-sharing may be mimicked by varying the number of points in the discrete part of the inputs.

Notation convention. Throughout the paper we adopt the following notation. Lower case variables are instances of upper case random variables (r.v.) which take on values in calligraphic alphabets. We let

$$\mathsf{N_d}(x) := \left\lfloor \sqrt{1+x} \right\rfloor, \tag{6}$$
$$\mathsf{I_g}(x) := \frac{1}{2}\log(1+x), \tag{7}$$
$$\mathsf{I_d}(X) := \left[H(X) - \mathsf{gap}_{(8)}\right]^+, \tag{8}$$
$$\mathsf{gap}_{(8)} := \frac{1}{2}\log\left(\frac{2\pi \mathrm{e}}{12}\right) + \frac{1}{2}\log\left(1 + \frac{12}{d^2_{\min(X)}}\right), \tag{9}$$

where the subscripts d and g remind the reader that discrete and Gaussian, respectively, inputs are involved. $H(X)$ is the entropy of the discrete r.v. X and $[x]^+ := \max\{0, x\}$. $[n_1 : n_2]$ is the set of integers from n_1 to $n_2 \ge n_1$. If A is a r.v. we denote its support by $\mathsf{supp}(A)$. The symbol $|\cdot|$ denotes: $|\mathcal{A}|$ is the cardinality of the set $\mathcal{A}$, $|X|$ is the cardinality of $\mathsf{supp}(X)$ of the random variable X, or $|x|$ is the absolute value of the real-valued x. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the largest integer not greater than x. $d_{\min(\mathcal{S})} := \min_{i \ne j : s_i, s_j \in \mathcal{S}} |s_i - s_j|$ is the minimum distance among the points in the set $\mathcal{S}$. With some abuse of notation we also use $d_{\min(X)}$ to denote $d_{\min(\mathsf{supp}(X))}$ for a r.v. X. $X \sim \mathsf{PAM}\left(N, d_{\min(X)}\right)$ denotes the uniform probability mass function over a zero-mean Pulse Amplitude Modulation (PAM) constellation with $|\mathsf{supp}(X)| = N$ points, minimum distance $d_{\min(X)}$, and average energy $\mathbb{E}[X^2] = d^2_{\min(X)} \frac{N^2-1}{12}$. $m(\mathcal{S})$ denotes Lebesgue measure of the set $\mathcal{S}$. For $i \in [1:2]$ we let $i' \in [1:2]$ to be $i' \ne i$.

II. Main Tools

In the rest of the paper, due to space limitations, proofs are omitted and may be found in [8]. At the core of our proofs is the following lower bound on the rate achieved by a discrete input on a point-to-point additive noise channel.

Prop. 1. *Let X_D be a discrete random variable, whose support has size N, minimum distance $d_{\min(X_D)}$, and average power $\mathbb{E}[X_D^2]$. Let Z be a zero-mean unit-variance random variable independent of X_D (not necessarily Gaussian). Then*

$$\mathsf{I_d}(X_D) \le I(X_D; X_D + Z) \le H(X_D), \tag{10}$$

where $\mathsf{I_d}(\cdot)$ is defined in (8).

Rem. 1. *The upper bound in* (10) *for $Z = Z_G \sim \mathcal{N}(0,1)$ may be tightened to*

$$I(X_D; X_D + Z_G) \le \min\left(H(X_D), \mathsf{I_g}(\mathbb{E}[X_D^2])\right), \tag{11}$$

since a Gaussian input is capacity achieving for the power-constrained point-to-point Gaussian noise channel.

In the following we shall use a mixed input at each user; this implies that a receiver sees a linear combination of two discrete constellations; in order to apply Prop. 1 we need to lower bound the minimum distance of the resulting sum-set. The following set of sufficient conditions will play an important role in evaluating the minimum distance of mixed input constellations in our TIN inner bound.

Prop. 2. *Let $(h_x, h_y) \in \mathbb{R}^2$ be two constants. Let $X \sim \mathsf{PAM}(|X|, d_{\min(X)})$ and $Y \sim \mathsf{PAM}(|Y|, d_{\min(Y)})$. Then*

$$|h_x X + h_y Y| = |X||Y|,$$
$$d_{\min(h_x X + h_y Y)} = \min\left(|h_x| d_{\min(X)}, |h_y| d_{\min(Y)}\right),$$

under the following conditions

$$\textit{either}\ \ |Y||h_y| d_{\min(Y)} \le |h_x| d_{\min(X)} \tag{12a}$$
$$\textit{or}\ \ |X||h_x| d_{\min(X)} \le |h_y| d_{\min(Y)}. \tag{12b}$$

When Prop. 2 is not applicable we shall use:

Prop. 3. *Let* $X \sim \mathsf{PAM}(|X|, d_{\min(X)})$ *and* $Y \sim \mathsf{PAM}(|Y|, d_{\min(Y)})$*. Then for* $(h_x, h_y) \in \mathbb{R}^2$ *we have*

$$|h_x X + h_y X| = |X||Y| \text{ almost surely (a.s.)} \tag{13a}$$

and for any $\gamma > 0$ *there exists a set* $E \subseteq \mathbb{R}$ *such that for all* $(h_x, h_y) \in E$ *we have*

$$d_{\min(h_x X + h_y Y)} \geq \kappa_{\gamma,|X|,|Y|} \min \Bigg(|h_x| d_{\min(X)}, \\ |h_y| d_{\min(Y)}, \max\left(\frac{|h_x| d_{\min(X)}}{|Y|}, \frac{|h_y| d_{\min(Y)}}{|X|} \right) \Bigg), \tag{13b}$$

where the complement of the set E*, referred to as the "outage set," has Lebesgue measure smaller than* γ*, and where*

$$\kappa_{\gamma,|X|,|Y|} = \frac{\gamma}{2(1 + \ln(\max(|X|, |Y|)))}.$$

III. Main Result

For the G-IC in (1) we now evaluate the TIN region in (5) with mixed inputs

$$X_i = \sqrt{1-\delta_i}\ X_{iD} + \sqrt{\delta_i}\ X_{iG},\ i \in [1:2]: \tag{14a}$$

$$X_{iD} \sim \mathsf{PAM}\left(N_i, \sqrt{\frac{12}{N_i^2 - 1}} \right), \tag{14b}$$

$$X_{iG} \sim \mathcal{N}(x_{iG}; 0, 1), \tag{14c}$$

$$\mathbf{P} := [N_1, N_2, \delta_1, \delta_2] \in \mathbb{N} \times \mathbb{N} \times [0,1] \times [0,1], \tag{14d}$$

where the random variables X_{ij} are independent for $i \in [1:2],\ j \in \{D, G\}$. A careful choice of the parameters in $\mathbf{P}$ will lead to the desired results in different parameter regimes. From the TIN region in (5) we have:

Prop. 4. *For the G-IC the following region is achievable*

$$\mathcal{R}_{in}(\mathsf{S}, \mathsf{I}) := \bigcup_{\mathbf{P} \in \mathbb{N}^2 \times [0,1]^2} \mathcal{R}_{in}(\mathsf{S}, \mathsf{I}; \mathbf{P}), \tag{15a}$$

where $\mathcal{R}_{in}(\mathsf{S}, \mathsf{I}; \mathbf{P})$ *is a lower bound on the TIN region in* (5) *evaluated for the mixed input in* (14) *with fixed parameter vector* $\mathbf{P} := [N_1, N_2, \delta_1, \delta_2]$*, given by*

$$\mathcal{R}_{in}(\mathsf{S}, \mathsf{I}; \mathbf{P}) := \Bigg\{ (R_1, R_2) : R_i \leq \mathsf{I_d}(S_i) + \mathsf{I_g}\left(\frac{\mathsf{S}\delta_i}{1 + \mathsf{I}\delta_{i'}} \right) \\ - \min\left(\log(N_{i'}), \mathsf{I_g}\left(\frac{\mathsf{I}(1-\delta_{i'})}{1 + \mathsf{I}\delta_{i'}} \right) \right), i \in [1:2] \Bigg\}, \tag{15b}$$

and where the equivalent discrete constellations seen at the receivers are

$$S_i := \frac{\sqrt{1-\delta_i}\sqrt{\mathsf{S}} X_{iD} + \sqrt{1-\delta_{i'}}\sqrt{\mathsf{I}} X_{i'D}}{\sqrt{1 + \mathsf{S}\delta_i + \mathsf{I}\delta_{i'}}}, \tag{15c}$$

for $i' \neq i \in [1:2]$ *and the functionals* $\mathsf{I_d}(\cdot)$ *and* $\mathsf{I_g}(\cdot)$ *are defined in* (8) *and* (7)*, respectively.*

From Prop. 4, it follows that the following gDoF region is achievable:

Prop. 5. *Parametrize the number of points and the power splits in the mixed inputs in* (14) *as*

$$N_i = \mathsf{N_d}(\mathsf{S}^{\beta_i}),\ \beta_i \geq 0,\quad \delta_i = \frac{1}{1 + \mathsf{S}^{p_i}},\ p_i \geq 0, \tag{16a}$$

for $i \in [1:2]$*, where* $\mathsf{N_d}(\cdot)$ *is defined in* (6)*. For the G-IC an achievable gDoF region a.e. is*

$$\mathcal{D}_{in}(\alpha) = \bigcup_{\mathbf{p} := [\beta_1, \beta_2, p_1, p_2], \mathbf{p} \in \mathbb{R}_+^4} \mathcal{D}_{in}(\alpha; \mathbf{p}), \tag{16b}$$

$$\mathcal{D}_{in}(\alpha; \mathbf{p}) = \Big\{ (d_1, d_2) : d_i \leq \beta_1 + \beta_2 + [1 - p_i - [\alpha - p_{i'}]^+]^+ \\ - \min(\beta_{i'}, \alpha - [\alpha - p_{i'}]^+),\ i \in [1:2]) \Big\} \tag{16c}$$

provided that at least one of the following conditions hold for all $i \in [1:2]$

$$1)\ 1 \leq \alpha - \beta_{i'} \text{ and } \max(1 - p_i, \alpha - p_{i'}, 0) \\ - \min(1 - \beta_i, \alpha - \beta_{i'}) = 0, \tag{16d}$$

$$2)\ \alpha \leq 1 - \beta_i \text{ and } \max(1 - p_i, \alpha - p_{i'}, 0) \\ - \min(1 - \beta_i, \alpha - \beta_{i'}) = 0, \tag{16e}$$

$$3)\ \max(1 - p_i, \alpha - p_{i'}, 0) - \min(1 - \beta_i, \alpha - \beta_{i'}, \\ \max(1 - \beta_i - \beta_{i'}, \alpha - \beta_i - \beta_{i'})) = 0. \tag{16f}$$

The main result of this paper is the following theorem:

Theorem 1. *For the symmetric G-IC the TIN-based achievable region in* (15a) *is optimal to within a constant gap, or a gap of order* $O(\log\log(\mathsf{S}))$ *up to an outage set, and it is therefore gDoF optimal up to a set of measure zero.*

Proof: Depending on the parameters ($\mathsf{S}, \mathsf{I} = \mathsf{S}^\alpha$) we identify the following operational regimes:

1) Very Strong Interference: $\alpha \geq 2$.
2) Strong Interference: $1 \leq \alpha < 2$.
3) Weak Interference Type I: $2/3 \leq \alpha < 1$.
4) Weak Interference Type II: $1/2 \leq \alpha < 2/3$.
5) Very Weak Interference: $0 \leq \alpha < 1/2$.

We shall consider each regime individually by picking the parameters of the mixed inputs such that inner and outer bounds approximately match. Here we view the discrete and the Gaussian part of the mixed input as a "common" and a "private" message, respectively, in the HK scheme. Thus, a critical aspect of our analysis is to identify the optimal rate split for each point on the convex closure of the capacity region outer bound. By using [9, Lem. 4], which can be thought of as the gDoF region in (4) before Fourier-Motzkin elimination, we found that either it is optimal to user $p_i = \alpha$ in (16a), or equally split the rate between common and private messages.

a) Very Strong Interference Regime: $\alpha \geq 2$*:* In this regime it is optimal to send only common messages, and to perform successive decoding starting with the interfering message. While joint decoding is not allowed in our TIN region, the discrete part of the inputs behaves as common message (i.e., as if it could be decoded at non intended destination). The gDoF region in (4) is

$$\mathcal{D}^{\text{Very Strong}} : d_1 \leq 1,\ d_2 \leq 1,$$

which is easily matched by the achievable scheme in Prop. 5 by using

$$\mathbf{p} = [1, 1, \infty, \infty].$$

In order to complete the proof we show that the condition in (16d) is satisfied:

$$1 \le \alpha - \beta_i, \ \forall i \in [1:2] \Leftrightarrow 1 \le \alpha - \beta \Leftrightarrow 2 \le \alpha,$$

where the last implication comes from the definition of regime.

For the finite (S, I), the capacity region in $\max(R_1, R_2) \le \mathsf{I_g}(\mathsf{S})$. We set $\delta_i = 0, \ N_i = \mathsf{N_d}(\mathsf{S}), i \in [1:2]$, and thus for sum-sets in (15c) we have $S_1 = S_2 := S$; the gap is thus readily computed as

$$\mathsf{gap} = \mathsf{I_g}(\mathsf{S}) - \mathsf{I_d}(S) \le \frac{1}{2}\log\left(\frac{16\pi \mathrm{e}}{12}\right) \approx 1.75.$$

b) Strong Interference: $1 \le \alpha < 2$*:* Capacity in this regime is achieved by sending only common messages. Thus, similar to the very strong interference regime, we set $\delta_1 = \delta_2 = 0$. However, unlike in the very strong interference regime, we vary the number of points in the discrete part in order to mimic time sharing, for a reason that will become clear soon. In this regime the gDoF region is that of a Compound MAC and in parametric form is given by

$$\mathcal{D}^{\text{Strong}}(\alpha) = \bigcup_{t \in [0,1]} \left\{ \begin{array}{l} d_1 \le t + (1-t)(\alpha - 1) := \beta_1(t) \\ d_2 \le t(\alpha - 1) + (1-t) := \beta_2(t) \end{array} \right\}.$$

Note that the parameter $t \in [0,1]$ is used to time-share between the two corner points of the capacity region (in this case a MAC-region). This parametric form of the gDoF region is matched to the achievable scheme in Prop. 5 by using

$$\mathbf{p}(t) = [\beta_1(t), \beta_2(t), \infty, \infty], \ \forall t \in [0,1].$$

To complete the proof we choose to verify the condition in (16f) for $i = 1$ and for all $t \in [0,1]$ (by symmetry, it also holds for $i = 2$); we have

$$\begin{aligned} &\min(1 - \beta_1(t), \alpha - \beta_2(t), \max(1 - \beta_1(t) - \beta_2(t), \\ &\quad \alpha - \beta_1(t) - \beta_2(t))), \\ &= \max(1 - t - (1-t)(\alpha - 1), \alpha - t(\alpha - 1) - (1-t), 0), \\ &= 0, \ \forall t \in [0,1], \end{aligned}$$

and the last equality follows since in this regime we have $\alpha \le \beta_1(t), \ \alpha \le \beta_2(t), \ \forall t \in [0,1]$.

For finite (S, I) the outer bound for this regime can be written as

$$\mathcal{R}^{\text{Strong}}(\mathsf{S}, \mathsf{I}) = \bigcup_{t \in [0,1]} \left\{ \begin{array}{ll} R_1 & \le (1-t)\mathsf{I_g}\left(\frac{\mathsf{I}}{1+\mathsf{S}}\right) + t\mathsf{I_g}(\mathsf{S}) \\ & := \mathsf{I_g}(\mathsf{S}_{1,t}) \\ R_2 & \le t\mathsf{I_g}\left(\frac{\mathsf{I}}{1+\mathsf{S}}\right) + (1-t)\mathsf{I_g}(\mathsf{S}) \\ & := \mathsf{I_g}(\mathsf{S}_{2,t}) \end{array} \right\}.$$

By choosing $\delta_i = 0, \ N_i = \mathsf{N_d}(\mathsf{S}_{i,t}), i \in [1:2]$, we have that the gap is

$$\begin{aligned} \mathsf{gap} &= \max_{i \in [1:2]} \left(\mathsf{I_g}(\mathsf{S}_{i,t}) - \mathsf{I_d}(\mathsf{S}_{i,t})\right) \\ &\le \frac{1}{2}\log\left(\frac{2\pi \mathrm{e}}{3}\left(1 + \frac{8\left(1 + \ln(\sqrt{1+\mathsf{S}})\right)^2}{\gamma^2}\right)\right), \end{aligned}$$

where we used Prop. 3 to bound minimum distance of the sum-set constellations at the receivers and where such a bound holds everywhere except a set of measure γ.

c) Weak Type I: $2/3 \le \alpha < 1$*:* In this regime the best known strategy is to send common and private messages with a power split as in [6]. Thus, as in the strong interference regime, we vary the number of points in the discrete parts to mimic time sharing, but unlike the strong interference regime, we also use the Gaussian part of the inputs (i.e., δ_1, δ_2 are non-zero) to allow for a private message. As we shall see, for our TIN-based scheme a single power split as in [6] does not suffice to achieve all points in the gDoF region; we will thus also vary δ_1, δ_2 in addition to β_1 and β_2. In this regime the gDoF region in (4) can be written in the parametric form as

$$\begin{aligned} \mathcal{D}^{\text{Weak I}}(\alpha) = \bigcup_{t \in [0,1]} \Big(&\mathcal{D}_{d_1+d_2}(\alpha, t) \cup \mathcal{D}_{2d_1+d_2}(\alpha, t) \\ &\cup \mathcal{D}_{d_1+2d_2}(\alpha, t)\Big), \end{aligned}$$

$$\mathcal{D}_{d_1+d_2}(\alpha, t) = \left\{ \begin{array}{ll} d_1 & \le t(2\alpha - 1) + (1-t)(1-\alpha) \\ & \quad +1 - \alpha := \beta_{1,a}(t) + 1 - \alpha \\ d_2 & \le t(1-\alpha) + (1-t)(2\alpha - 1) \\ & \quad +1 - \alpha := \beta_{2,a}(t) + 1 - \alpha \end{array} \right\},$$

$$\mathcal{D}_{2d_1+d_2}(\alpha, t) = \left\{ \begin{array}{ll} d_1 & \le t(2\alpha - 1) + (1-t)\alpha + 1 - \alpha \\ & := \beta_{2,a}(t) + 1 - \alpha \\ d_2 & \le t(1-\alpha) + t(1-\alpha) \\ & := \beta_{2,b}(t) + t(1-\alpha) \end{array} \right\},$$

$$\mathcal{D}_{d_1+2d_2}(\alpha, t) = \left\{ \begin{array}{ll} d_1 & \le \beta_{2,b}(t) + t(1-\alpha) \\ d_2 & \le \beta_{2,a}(t) + 1 - \alpha \end{array} \right\},$$

where again $t \in [0,1]$ is used to time-share.

Next, $\mathcal{D}_{d_1+d_2}(\alpha, t)$ is matched to Prop. 5 by picking

$$\mathbf{p}(t) = [\beta_{1,a}(t), \beta_{1,b}(t), \alpha, \alpha], \ \forall t \in [0,1].$$

To complete the achievability of $\mathcal{D}_{d_1+d_2}(\alpha, t)$ we show that (16f) is satisfied. Due to the symmetry it is enough to verify condition (16f) for $i = 1$ for all $t \in [0,1]$, which gives

$$\begin{aligned} &1 - \alpha - \min(1 - \beta_{1,a}(t), \alpha - \beta_{2,a}(t), \\ &\max(1 - \beta_{1,a}(t) - \beta_{2,a}(t), \alpha - \beta_{1,a}(t) - \beta_{2,a}(t))), \\ &= 1 - \alpha - \min(1 - t(2\alpha - 1) - (1-t)(1-\alpha), \\ &\quad \alpha - t(1-\alpha) - (1-t)(2\alpha - 1), 1 - \alpha), \ \forall t \in [0,1], \\ &= 1 - \alpha - (1 - \alpha) = 0, \ \forall t \in [0,1]. \end{aligned}$$

The region $\mathcal{D}_{2d_1+d_2}(\alpha, t)$ is matched to Prop. 5 with

$$\mathbf{p}(t) = [\beta_{2,a}(t), \beta_{2,b}(t), \alpha, 1 - t(1-\alpha)], \ \forall t \in [0,1].$$

To complete achievability of $\mathcal{D}_{2d_1+d_2}(\alpha, t)$, we verify condition (16f) for S_1 (the one for S_2 follows similarly), that is

$$\begin{aligned}
&\max(0, 1-p_1, \alpha-p_2) - \min(1-\beta_{2,a}(t), \alpha-\beta_{2,b}(t),\\
&\max(1-\beta_{2,a}(t)-\beta_{2,b}(t), \alpha-\beta_{2,a}(t)-\beta_{2,b}(t)))\\
&= \max(0, 1-\alpha, \alpha-(1-t(1-\alpha)))\\
&- \min(1-(t(2\alpha-1)+(1-t)\alpha), \alpha-t(1-\alpha), 1-\alpha)\\
&= 0, \ \forall t \in [0,1].
\end{aligned}$$

In order to achieve $\mathcal{D}_{2d_1+d_2}$ and $\mathcal{D}_{d_1+2d_2}$ (i.e., points on the closure on the region but not on the sum-rate face), we had to vary both the common (i.e β_i's) and private (i.e. p_i's) part of the messages to mimic time sharing and power control. Another important observation is that we exactly characterized the set of parameters needed to achieve every rate pair on the closure of $\mathcal{D}^{\text{Weak I}}$ – a characterization that is not obvious from the work in [6], which in fact seems to suggest that a single power split suffices to achieve all points in the gDoF region.

The proof of optimality of the proposed scheme to within a gap of order $O(\log\log(\mathsf{S}))$ is quite tedious and it is not reported here for sake of space; the details can be found in [8]).

Rem. 2. *We remark that in order to bound the minimum distance of the received constellations in strong and weak type I interference regimes we have used Prop. 3, which implies that there exists a set of measure* 0 *where* (15a) *is strictly below the outer bound* (4)*. This comes as no surprise, from [10] we know that in the vicinity of* $\alpha = 1$*, for two-user G-IC, constellation based schemes perform poorly when channel gains are rationally dependent and outage sets have been used to circumvent this. Moreover, for more than two users, this becomes an artifact of the capacity region itself.*

d) Weak Interference Type II: $1/2 \le \alpha < 2/3$*:* In this regime the gDoF region in (4) can be written as

$$\mathcal{D}^{\text{Weak II:}}(\alpha) = \bigcup_{t\in[0,1]} \Big(\mathcal{D}^{H}_{d_1+d_2}(\alpha,t) \cup \mathcal{D}^{L}_{d_1+d_2}(\alpha,t) \cup \mathcal{D}_{2d_1+d_2}(\alpha,t) \cup \mathcal{D}_{d_1+2d_2}(\alpha,t)\Big),$$

$$\mathcal{D}_{d_1+d_2}(\alpha,t) = \left\{\begin{array}{ll} d_1 & \le (1-t)(2-2\alpha) \\ & +t(4\alpha-2) := \beta_{1,a}(t) \\ & +(1-t)(1-\alpha)+t(2\alpha-1) \\ d_2 & \le (1-t)(4\alpha-2) \\ & +t(2-2\alpha) := \beta_{1,b}(t) \\ & +(1-t)(2\alpha-1)+t(1-\alpha) \end{array}\right\},$$

$$\mathcal{D}_{2d_1+d_2}(\alpha,t) = \left\{\begin{array}{ll} d_1 & \le (1-t)(2-2\alpha)+t \\ & := \beta_{2,a}(t)+1-\alpha \\ d_2 & \le (1-t)(4\alpha-2) \\ & := \beta_{2,b}(t)+(1-t)(2\alpha-1) \end{array}\right\}$$

$$\mathcal{D}_{d_1+2d_2}(\alpha,t) = \left\{\begin{array}{ll} d_1 & \le \beta_{2,b}(t)+(1-t)(2\alpha-1) \\ d_2 & \le \beta_{2,a}(t)+1-\alpha \end{array}\right\}.$$

Next, $\mathcal{D}_{d_1+d_2}(\alpha,t)$ is achievable for all $t \in [0,1]$ by picking

$$\mathbf{p}(t) = [\beta_{1,a}(t), \beta_{1,b}(t), 1-\beta_{1,a}(t), 1-\beta_{1,b}(t)],$$

in Prop. 5 and verifying the condition in (16e)–which we do not report here as it is similar to the previous regime. For $\mathcal{D}_{d_1+2d_2}(\alpha,t)$ we pick

$$\mathbf{p}(t) = [\beta_{2,a}(t), \beta_{2,b}(t), \alpha, 1-\beta_{2,b}(t)],$$

in Prop. 5 and verify the condition (16d) for receiver one and condition (16e) for receiver two. By symmetry, $\mathcal{D}_{d_1+2d_2}(\alpha,t)$ is achievable by swapping the users.

The proof of optimality of the proposed scheme to within a constant gap is quite tedious and it is not reported here for sake of space; the details can be found in [8]).

e) Very Weak Interference: $0 \le \alpha < 1/2$*:* Gaussian inputs with power control and treating interference as noise are optimal to within 1/2 bit from the work of [6]. Since this scheme is a special case of the TIN region with mixed inputs, with $N_1 = N_2 = 1$, the claimed optimality follows. ∎

IV. Conclusion

In this paper we proved that a very simple, generally applicable, lower bound that does neither require joint decoding nor time sharing is optimal to within an additive gap (either constant uniformly over the channel gains, or of order $O(\log\log(\mathsf{S}))$ up to an outage set of controllable measure) and thus achieves the optimal gDoF region of the two-user symmetric G-IC for all channel parameters. Our result demonstrates that properly accounting for the distribution of the interference (i.e., not Gaussian with our mixed inputs) when treating interference as noise results in near optimal rates in all channel parameters. Moreover, an exact characterization of the closure of gDoF region, together with the fact that we used PAM signals with closed form expressions for the number of points and the power splits, makes the scheme practical.

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