ON THE MUTUAL INFORMATION OF TIME REVERSAL FOR NON-STATIONARY CHANNELS

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ABSTRACT

Time-reversal (TR) techniques have been shown to lead to gains in detection and enable super-resolution focusing. These gains have thus far mainly been demonstrated for time invariant channels, where the channel remains constant between the initial and time-reversed signal transmissions. Here, we are interested in determining whether TR techniques may be beneficial in time-varying channels. We approach this problem by comparing the mutual information between the channel and the received signals over two time slots of a radar system which does / does not use TR. Besides evaluating this mutual information, and showing that for this setup it is equal to directed information which might be of interest in feedback-like channels, we also provide a low-rank interpretation of this mutual information for Gaussian channels. Numerical evaluations suggest that if the channels are non-stationary yet correlated, TR may still provide information gains over non time-reversed systems.

Index Terms— Time reversal, Radar, Mutual Information, Stochastic time-varying channel, non-stationary channel

1. INTRODUCTION

We consider monostatic radar-based time reversal (TR) over time-varying channels. In TR, a signal is radiated; the backscattered signal is then recorded, time-reversed, energy scaled and re-transmitted. TR may lead to super-resolution spatio-temporal focusing using multiple antennas, and detection gains for single and multiple antennas [1–6]. Most of these contributions in TR assumed the channel to be invariant from the initial signal transmission to the time-reversed re-transmission. The question of whether time-reversal is beneficial in time-varying channels remains. We make analytical progress towards this by introducing and analyzing the *information gain* (or *mutual information*) in TR systems as compared to conventional systems for time-varying channels. The TR system uses the channel twice: a single antenna is assumed to first probe the channel, and then uses the channel to transmit the time-reversed received signal. The non-TR counterpart denoted as "conventional" system probes the channel twice using the same signal. The channels are assumed to be linear, stochastic, subject to additive Gaussian noise, and time-varying.

Contributions and organization. We introduce the channel models for both TR and conventional use in Section 2. We then introduce the *information gain* metric, or relevant mutual information quantity for these models in Section 3, before analytically comparing the difference in information gains for TR and conventional channels in Section 4. Furthur , we demonstrate that the mutual information is equal to the directed information, and provide a low-rank interpretation of this metric. Finally, in Section 5 we numerically evaluate the difference in information gain of the TR and conventional channel. Our central contributions are 1) addressing TR from an analytical and information theoretic perspective for the first time, and 2) using this framework to quantitatively analyze the impact of time-varying channels on TR as compared to conventional systems.

Related work. TR has been studied for invariant channels time-reversal in optics, ultrasound and acoustics, radar and communications, in for example [1–6] and references therein. Work on TR in time-varying channels is limited: it was acknowledged in [4, pg. 36-37] via experimental insights that TR focusing degrades in non-stationary time-varying environments such as the time-varying ocean surface and its volume. In communication applications, it was experimentally shown that time varying channels affect the TR performance, in [7, 8] –interestingly the conclusions drawn are similar to those drawn here – that TR may still be beneficial in channels which are correlated but not necessarily identical over time.

2. CHANNEL MODEL

We outline the two assumed transmission stages of the TR and conventional channel models. Other TR protocols with more than two stages may be devised, but for simplicity, we initially limit ourselves to two stages.

Stage 1: The P transmitted baseband samples are $\mathbf{s} := [s(0), s(1), \ldots, s(P-1)]^T$, and the matrix $\mathbf{S} \in \mathbb{C}^{N\times M}$ denotes the convolution matrix comprised of the samples $\mathbf{s}$. The received signal at $\mathbf{y_1}$ is then given by the N complex samples

$$\mathbf{y_1} = \mathbf{S}\boldsymbol{\alpha_1} + \mathbf{v_1}, \tag{1}$$

where $\mathbf{v}_1 := [v_1(0), v_1(1), \ldots, v_1(N-1)]^T$, is the additive noise and $\boldsymbol{\alpha_1} = [\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1M}]^T \in \mathbb{C}^M$ is the sampled channel impulse response.

Stage 2: The TR system transmitter then transmits the scaled, by $\kappa = \frac{||\mathbf{s}||}{||\bar{\mathbf{y}}_1||}$, time-reversed channel output $\mathbf{y}_1$. The convolution matrix of the time-reversed output $\mathbf{y_1}$ is denoted by $\bar{\mathbf{Y}}_\mathbf{1} \in \mathbb{C}^{N\times M}$. In this stage, the channel impulse response is given by $\boldsymbol{\alpha_2} \in \mathbb{C}^M$ and the new noise is $\mathbf{v_2}$, which need not follow the same statistics as in the first stage. In the "conventional" channel model, the same waveform $\mathbf{s}$ is sent during the second stage. Thus, the second stage outputs of the TR and conventional system, are given, respectively,

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by

$$\mathbf{y_2^{TR}} := \kappa \bar{\mathbf{Y}}_1 \boldsymbol{\alpha}_2 + \mathbf{v_2} \qquad (TR) \tag{2}$$

$$\mathbf{y_2^{nTR}} := \mathbf{S}\boldsymbol{\alpha}_2 + \mathbf{v_2} \qquad (No\ TR). \tag{3}$$

Assumptions: For analytical tractability, we assume that $\boldsymbol{\alpha} := [\boldsymbol{\alpha_1}^T, \boldsymbol{\alpha_2}^T]^T$ is jointly Gaussian with probability density function (pdf), of mean and covariance, respectively,

$$\boldsymbol{\mu} := [\boldsymbol{\mu_1}^T, \boldsymbol{\mu_2}^T]^T, \qquad \mathbf{C} = \begin{bmatrix} \mathbf{C_1} & \mathbf{C_{12}} \\ \mathbf{C_{12}}^H & \mathbf{C_2} \end{bmatrix}. \tag{4}$$

In (4), $\mathbf{C_i} = \text{Cov}\{\boldsymbol{\alpha}_i, \boldsymbol{\alpha}_i\}, \mathbf{i} = 1, 2$ and $\mathbf{C_{12}} = \text{Cov}\{\boldsymbol{\alpha_1}, \boldsymbol{\alpha_2}\}$, where $\text{Cov}\{\mathbf{x}, \mathbf{y}\}$ is the covariance (matrix) between vectors $\mathbf{x}$ and $\mathbf{y}$. The concatenated noise vector, $\mathbf{v} = [\mathbf{v_1}^T, \mathbf{v_2}^T]^T$ is independent of $\boldsymbol{\alpha}$ and is assumed to be zero-mean, jointly Gaussian with corresponding covariance matrix, given by,

$$\mathbf{V} = \begin{bmatrix} \mathbf{V_1} & \mathbf{V_{12}} \\ \mathbf{V_{12}}^H & \mathbf{V_2} \end{bmatrix}. \tag{5}$$

where $\mathbf{V_i} = \text{Cov}\{\mathbf{v_i}, \mathbf{v_i}\}, \mathbf{i} = 1, 2$ and $\mathbf{V_{12}} = \text{Cov}\{\mathbf{v_1}, \mathbf{v_2}\}$.

3. INFORMATION GAIN / MUTUAL INFORMATION

We quantify the utility of TR channels using "information gain", or the mutual information between the appropriate channel input/output quantities as a metric. In radar channels, information theoretic metrics such as mutual information and conditional entropy have been used for a variety of purposes including waveform design [9, 10], waveform scheduling [11, 12], and sensor management [13, 14]. We are motivated to use mutual information or information gain as metric for the same purpose as the aforementioned work – as a surrogate metric, which is not linked to a specific task such as detection, to quantify the amount of information gained via TR.

The radar system wishes to learn about the channel $\boldsymbol{\alpha}$ from the received signals $\mathbf{y} = [\mathbf{y_1}^T\ \mathbf{y_2}^{(\mathbf{TR})T}]^T$ (TR) or $\mathbf{y} = [\mathbf{y_1}^T\ \mathbf{y_2}^{(\mathbf{nTR})T}]^T$ (conventional), given knowledge of the transmitted waveforms $\mathbf{s}$. Define the "information gain" about $\boldsymbol{\alpha}$ from $\mathbf{y} := [\mathbf{y_1}^T\ \mathbf{y_2}^T]^T$ given knowledge of the transmitted waveform $\mathbf{s}$ by the mutual information (MI) between $\boldsymbol{\alpha}$ and $\mathbf{y}$ given $\mathbf{s}$, denoted by $I(\boldsymbol{\alpha}; \mathbf{y}|\mathbf{s}) := \mathbb{E}_{p(\boldsymbol{\alpha},\mathbf{y},\mathbf{s})}\{\ln(\frac{p(\boldsymbol{\alpha},\mathbf{y}|\mathbf{s})}{p(\boldsymbol{\alpha}|\mathbf{s}),p(\mathbf{y}|\mathbf{s})})\} = h(\boldsymbol{\alpha}|\mathbf{s}) - h(\boldsymbol{\alpha}|\mathbf{s}, \mathbf{y})$. Here $h(\cdot|\cdot)$ is defined as the conditional differential entropy, $p(\cdot|\cdot)$ is the conditional pdf, and $\mathbb{E}$ denotes the expectation operator. Recall the following [15]:

$$h(\boldsymbol{\alpha}|\mathbf{s}) := -\mathbb{E}_{p(\boldsymbol{\alpha},\mathbf{s})}\{\ln(p(\boldsymbol{\alpha}|\mathbf{s}))\} \tag{6}$$

$$h(\boldsymbol{\alpha}|\mathbf{s}, \mathbf{y}) = -\mathbb{E}_{p(\boldsymbol{\alpha},\mathbf{s},\mathbf{y})}\{\ln(p(\boldsymbol{\alpha}|\mathbf{s}, \mathbf{y}))\}$$

Remark 1. *Time-Reversal: A channel with feedback or not?* In a mono-static radar system which employs TR and the next waveform transmitted depends on the previously received data, intuitively at least, TR appears analogous to an information theoretic channel where the encoder employs feedback (i.e. encoders have access to previous channel outputs and may let their subsequent channel inputs be functions of these outputs). For feedback channels, *directed information* (DI), rather than mutual information between them, is a more relevant metric (in terms of channel capacity) [16, 17]. In this case, the information gained might be captured by the following two-step causally conditioned DI:

$$DI(\mathbf{s}, \bar{\mathbf{y}}_1) = I(\boldsymbol{\alpha} \longrightarrow \mathbf{y}||\mathbf{s}, \bar{\mathbf{y}}_1) \tag{7}$$

$$:= I(\boldsymbol{\alpha_1}; \mathbf{y_1}|\mathbf{s}) + I(\boldsymbol{\alpha}; \mathbf{y_2}|\mathbf{s}, \bar{\mathbf{y}}_1, \mathbf{y_1}) \tag{8}$$

$$= I(\boldsymbol{\alpha_1}; \mathbf{y_1}|\mathbf{s}) + I(\boldsymbol{\alpha}; \mathbf{y_2}|\mathbf{s}, \mathbf{y_1}) \tag{9}$$

where (9) follows by substituting $\bar{\mathbf{y}}_1 = f(\mathbf{y_1}) = \bar{\mathbf{I}}\mathbf{y}_1^*$ in (8), where $\bar{\mathbf{I}}$ is a matrix with ones on the anti-diagonal, and zeros elsewhere. One may thus wonder why we are using MI instead of DI. Interestingly, for our Gaussian channel model, the DI and MI are equal, which at first is somewhat surprising (as often in channels where feedback is employed, DI is strictly less than MI). While this may be able to be shown by relating / extending the proof of [18][Proposition 3, part 3)] from continuous time to discrete time, and from scalar to vector observations, we prove it directly using linear algebra.

Theorem 1. *The Mutual information and directed information are equal for the TR channel in* (1),(2) *under the Gaussian assumptions in* (4), (5).

Proof. It suffices to show that $I(\boldsymbol{\alpha_2}; \mathbf{y_1}|\boldsymbol{\alpha_1}, \mathbf{s}) = h(\boldsymbol{\alpha_2}|\boldsymbol{\alpha_1}, \mathbf{s}) - h(\boldsymbol{\alpha_2}|\boldsymbol{\alpha_1}, \mathbf{y_1}, s) = 0$. Evaluating,

$$\begin{aligned} h(\boldsymbol{\alpha_2}|\boldsymbol{\alpha_1}, \mathbf{s}) &= h(\boldsymbol{\alpha_2}|\boldsymbol{\alpha_1}) = h(\boldsymbol{\alpha_1}, \boldsymbol{\alpha_2}) - h(\boldsymbol{\alpha_1}) \\ &= \ln\det(\mathbf{C}) - \ln\det(\mathbf{C_1}) \\ &= \ln\det(\mathbf{C_1}(\mathbf{C_2} - \mathbf{C_{12}}^H\mathbf{C}_1^{-1}\mathbf{C_{12}})) - \ln\det(\mathbf{C_1}) \\ &= \ln\det(\mathbf{C_2} - \mathbf{C_{12}}^H\mathbf{C}_1^{-1}\mathbf{C_{12}}) \end{aligned}$$

We wish to show that this is equal to the entropy below:

$$\begin{aligned} h(\boldsymbol{\alpha_2}|\boldsymbol{\alpha_1}, \mathbf{y_1}, s) &= \ln\det(\mathbf{E}(\mathbf{C_2} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}^H)) - \ln\det(\mathbf{E}) \\ &= \ln\det(\mathbf{C_2} - \mathbf{D}\mathbf{E}^{-1}\mathbf{D}^H)) \end{aligned} \tag{10}$$

where, using the Schur complement form for the partitioned matrix determinant,

$$\mathbf{D} = [\mathbf{C_{12}}^H, \mathbf{C_{12}}^H\mathbf{S}^H], \text{ and } \mathbf{E} = \begin{bmatrix} \mathbf{C_1} & \mathbf{C_1}\mathbf{S}^H \\ \mathbf{S}\mathbf{C_1} & \mathbf{S}\mathbf{C_1}\mathbf{S}^H + \mathbf{V_1} \end{bmatrix}.$$

The following may be readily derived:

$$\mathbf{E}^{-1} = \begin{bmatrix} (\mathbf{C_1} - \mathbf{C_1}\mathbf{S}^H(\mathbf{S}\mathbf{C_1}\mathbf{S}^H)^{-1}\mathbf{S}\mathbf{C_1})^{-1} & -\mathbf{S}^H\mathbf{V}_1^{-1} \\ -\mathbf{V}_1^{-1}\mathbf{S} & \mathbf{V}_1^{-1} \end{bmatrix}$$

$$(\mathbf{C_1} - \mathbf{C_1}\mathbf{S}^H(\mathbf{S}\mathbf{C_1}\mathbf{S}^H)^{-1}\mathbf{S}\mathbf{C_1})^{-1} = \mathbf{C}_1^{-1} + \mathbf{S}^H\mathbf{V}_1^{-1}\mathbf{S}.$$

Using the above equations and expanding the matrix product $\mathbf{D}\mathbf{E}^{-1}\mathbf{D}^H$, we can now show that $\mathbf{D}\mathbf{E}^{-1}\mathbf{D}^H = \mathbf{C_{12}}^H\mathbf{C}_1^{-1}\mathbf{C_{12}}$. Substituting this into (10), we have the proof □

That directed and mutual information are equal may be explained by the fact that we are interested in the DI between the *channel impulse response* $\boldsymbol{\alpha}$ and the output $\mathbf{y}$, conditioned on the channel inputs $\mathbf{s}, \bar{\mathbf{y}}_1$, and the channel impulse response does NOT employ feedback. Hence, because of the problem's application in learning about the channel rather than the input waveforms, what might appear to be a feedback channel does not result in different DI and MI.

4. TR VERSUS CONVENTIONAL CHANNELS

We consider the difference between the information gain achieved by the TR and the conventional channel:

$$\begin{aligned}\Upsilon := & I(\boldsymbol{\alpha_1},\boldsymbol{\alpha_2};\mathbf{y_1},\mathbf{y_2^{TR}}|\mathbf{s}) - I(\boldsymbol{\alpha_1},\boldsymbol{\alpha_2};\mathbf{y_1},\mathbf{y_2^{nTR}}|\mathbf{s}) \\ = & I(\boldsymbol{\alpha};\mathbf{y_2^{TR}}|\mathbf{s},\mathbf{y_1}) - I(\boldsymbol{\alpha};\mathbf{y_2^{nTR}}|\mathbf{s},\mathbf{y_1}),\end{aligned} \tag{11}$$

where (11) follows by the chain rule for mutual information [15], and the fact that $\mathbf{y_1}$ is the same regardless of whether TR is used or not. If $\Upsilon > 0$, we may conclude that TR yields more information about the channel $\boldsymbol{\alpha}$ than the conventional channel, and vice-versa if $\Upsilon < 0$.

The two terms in (11) may be evaluated, under our Gaussian assumptions, as (12) – (13). Although not explicitly shown, it is noted that κ is a function of $\mathbf{y_1}$ and stays inside the expectation operation. From (2), note that the pdf of $\mathbf{y_2^{TR}}$ given a $\mathbf{y_1}$ is normally distributed with a mean given by $\kappa\bar{\mathbf{Y}}_\mathbf{1}\mathbb{E}\{\boldsymbol{\alpha_2}\}$ and covariance matrix $\kappa^2\bar{\mathbf{Y}}_\mathbf{1}\mathbf{C_2}\bar{\mathbf{Y}}_\mathbf{1}^H + \mathbf{V_2}$, which yields (12). Unfortunately a closed form solution to (12) is not immediate, and hence Monte Carlo simulations (as employed in TR settings in e.g. [2, 3]) are employed here. To gain intuition, one may consider a simple scalar channel where we are able to evaluate the metric analytically, as shown in [19] but omitted due to space constraints.

4.1. A low rank / signal sub-space interpretation

Under some standard assumptions, we now analyze the less commonly used information gain metric from a low-rank subspace perspective with the goal of relating it to some more classical statistical signal processing concepts. In (12),and due to presence of noise in $\bar{\mathbf{Y}}_\mathbf{1}$, we may let rank$\{\bar{\mathbf{Y}}_\mathbf{1}\mathbf{C_2}\bar{\mathbf{Y}}_\mathbf{1}^H\} = M$. Likewise, rank of the positive definite noise covariance matrix, $\mathbf{V_2} = \sigma^2\mathbf{I}$ is $N > M$. Then, $\kappa^2\bar{\mathbf{Y}}_\mathbf{1}\mathbf{C_2}\bar{\mathbf{Y}}_\mathbf{1}^H + \mathbf{V_2}$ permits an eigen decomposition

$$\kappa^2\bar{\mathbf{Y}}_\mathbf{1}\mathbf{C_2}\bar{\mathbf{Y}}_\mathbf{1}^H + \sigma^2\mathbf{I} = [\mathbf{E_s}(\mathbf{y_1}),\mathbf{E_n}]\,\mathbf{D}\,[\mathbf{E_s}(\mathbf{y_1}),\mathbf{E_n}(\mathbf{y_1})]^H \tag{15}$$

where $[\mathbf{E_s}(\mathbf{y_1}),\mathbf{E_n}(\mathbf{y_1})] \in \mathbb{C}^{N\times N}$ is the unitary matrix whose whose columns are eigenvectors. In particular, $\kappa^2\bar{\mathbf{Y}}_\mathbf{1}\mathbf{C_2}\bar{\mathbf{Y}}_\mathbf{1}^H$ lies in the span of $\mathbf{E_s}(\mathbf{y_1})$, and likewise the columns in $\mathbf{E_n}(\mathbf{y_1}) \in \mathbb{C}^{N\times(N-M)}$ span the subspace orthogonal to $\mathbf{E_s}(\mathbf{y_1})$. The matrix D is diagonal consisting of the M eigenvalues, $\lambda_m(\mathbf{y_1}) + \sigma^2, m = 1, 2, \ldots, M$ and the rest of the eigenvalues are σ^2.

Further assume that $\mathbf{SC_2S}^H - (\mathbf{SC_{12}S}^H + \mathbf{V_{12}})^H(\mathbf{SC_1S}^H + \mathbf{V_1})^{-1}(\mathbf{SC_{12}S}^H + \mathbf{V_{12}})$ is rank deficient, with M non-zero eigenvalues, $\beta_m, m = 1, 2, \ldots, M$. We may then obtain the decomposition in (14) where, $\mathbf{F_s}$ and $\mathbf{F_n}$ have the same dimensions as $\mathbf{E_s}(\mathbf{y_1})$ and $\mathbf{E_n}(\mathbf{y_1})$, respectively. Then the matrix $\mathbf{SC_2S}^H - (\mathbf{SC_{12}S}^H + \mathbf{V_{12}})^H(\mathbf{SC_1S}^H + \mathbf{V_1})^{-1}(\mathbf{SC_{12}S}^H + \mathbf{V_{12}}) = span\{\mathbf{F_s}\}$. The matrix $\tilde{\mathbf{D}}$ is diagonal and consists of M corresponding eigenvalues $\beta_m + \sigma^2$, and the rest of the eigenvalues equal to σ^2.

Using (15) and (14), (11) can now be written as,

$$\begin{aligned}\Upsilon = & \mathbb{E}_{\mathbf{y_1}}\left\{\ln\left(\prod_{m=1}^{M}(\lambda_m(\mathbf{y_1}) + \sigma^2)\right)\right\} - \ln\left(\prod_{m=1}^{M}(\beta_m + \sigma^2)\right) \\ = & \sum_{m=1}^{M}\mathbb{E}_{\mathbf{y_1}}\left\{\ln\left(\frac{\lambda_m(\mathbf{y_1})}{\sigma^2} + 1\right))\right\} - \sum_{m=1}^{M}\ln\left(\frac{\beta_m}{\sigma^2} + 1\right)c\end{aligned} \tag{16}$$

The eigenvectors in $\mathbf{E_s}(\mathbf{y_1})$, $\mathbf{F_s}$ are derived from the spectral decomposition of the conditional covariance matrices (conditioned on $\mathbf{y_1}$) for the TR and conventional channel, respectively. While not the classical signal subspaces, we may define them as the *conditional "signal" subspaces*. Note that these subspaces contain the contributions of the noise from the first use of the TR channel $\bar{\mathbf{Y}}_\mathbf{1}$, as well as the noise covariance in $\mathbf{V_1}$ from the first use of the conventional channel, respectively (and hence are not necessarily strict "signal" subspaces). Regardless, one may view $\prod_{m=1}^{M}(\lambda_m(\mathbf{y_1})/\sigma^2 + 1)$ as a measure of the "*quality*" of the (noise corrupted) conditional signal subspace after TR for a given $\mathbf{y_1}$. In the same spirit, the same can be said about the term $\prod_{m=1}^{M}(\beta_m/\sigma^2 + 1)$, which measures the "*quality*" of the (noise corrupted) conditional signal subspace for the conventional channel. One may then interpret Υ in (16), and under these assumptions as comparing the quality of the expected conditional signal subspace after TR and the quality for the corresponding conditional subspace for the conventional channel.

5. SIMULATIONS

We consider a channel termed ch-$\mathcal{A}$ with covariance matrix $\mathbf{C}_\mathcal{A} = \left[\begin{smallmatrix}\mathbf{I} & \rho_a\mathbf{I} \\ \rho_a\mathbf{I} & \mathbf{I}\end{smallmatrix}\right]$. This implies that the M taps in $\boldsymbol{\alpha_i} = [\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{iM}]^T, i = 1, 2$ are uncorrelated within each stage but are correlated across stages, depending on the values assumed by ρ_a. To purely analyze the effects of the channel, we assume white noise, i.e. $\mathbf{V_{12}} = 0, \mathbf{V_i} = \mathbf{I}, i = 1, 2$. Two waveforms for the transmitted $\mathbf{s}$ are analyzed, the first is a BPSK symbol waveform comprising random ± 1, the other is a radar chirp waveform. The analysis is carried out in baseband. We let $M = 10$, and define the SNR as $|\mathbf{s}|^2 M/\sigma^2$. The number of Monte Carlo trials were set at 10,000 to evaluate the expectation operation.

In Fig.1, the value of Υ versus ρ_a are shown for the BPSK waveform and the chirp for ch-$\mathcal{A}$ at SNR=$0, 10, 20$dBs. In Fig. 1(a), for SNR=10,20dB, we see that the Υ is positive for high correlation ($\rho_a \in \pm(1, 0.5]$), indicating superior performance of the TR when compared to the conventional channel. For medium to low correlations, and not surprisingly, the opposite is true, i.e Υ becomes negative indicating that TR is not preferable when compared to using the channel conventionally. In particular, we see that for SNR=20dB, and for medium and low correlation, the metric assumes low values. Similar results are seen for the chirp waveform in Fig. 1(b). The break even points for Fig. 1(a) and Fig. 1(b), i.e. $\Upsilon = 0$ are different for the same SNR, hence a waveform dependency is also noted. The processing for the chirp was performed in the baseband bandwidth [2] which contains 99% of the signal energy. For the implementation, spectral content outside the band was notched, and an inverse FFT (IFFT) was employed to return to the time domain. Such frequency domain processing is not required for the BPSK, as it is wideband. Additional simulation results for a different covariance matrix $\mathbf{C}$ are provided in [19].

It is stressed that the metric Υ evaluates the TR and conventional channel on the "average". In other words, for low to medium correlation, we have seen instances where the difference between the MI between the TR and the conventional channels are actually positive, whereas on an average it is negative, i.e. $\Upsilon < 0$.

6. CONCLUSIONS

We have explored time-reversal in a two stage mono-static radar system from an information gain perspective. In particular, for time-

$$I(\boldsymbol{\alpha};\mathbf{y_2^{TR}}|\mathbf{s},\mathbf{y_1}) = h(\mathbf{y_2^{TR}}|\mathbf{s},\mathbf{y_1}) - h(\mathbf{y_2^{TR}}|\mathbf{y_1},\boldsymbol{\alpha_1},\boldsymbol{\alpha_2},\mathbf{s}) = \mathbb{E}_{\mathbf{y_1}}\left\{\ln\det(\kappa^2\bar{\mathbf{Y}}_\mathbf{1}\mathbf{C_2}\bar{\mathbf{Y}}_\mathbf{1}^H + \mathbf{V_2})\right\} - \ln\det(\mathbf{V_2}) \quad (12)$$

$$\begin{aligned} I(\boldsymbol{\alpha};\mathbf{y_2^{nTR}}|\mathbf{s},\mathbf{y_1}) &= h(\mathbf{y_2^{nTR}}|\mathbf{s},\mathbf{y_1}) - h(\mathbf{y_2^{nTR}}|\mathbf{s},\mathbf{y_1},\boldsymbol{\alpha_1},\boldsymbol{\alpha_2}) \quad (13)\\ &= \ln\det\left(\mathbf{SC_2S}^H + \mathbf{V_2} - (\mathbf{SC_{12}S}^H + \mathbf{V_{12}})^H(\mathbf{SC_1S}^H + \mathbf{V_1})^{-1}(\mathbf{SC_{12}S}^H + \mathbf{V_{12}}\right) - \ln\det(\mathbf{V_2}) \end{aligned}$$

$$\mathbf{SC_2S}^H + \sigma^2\mathbf{I} - (\mathbf{SC_{12}S}^H + \mathbf{V_{12}})^H(\mathbf{SC_1S}^H + \mathbf{V_1})^{-1}(\mathbf{SC_{12}S}^H + \mathbf{V_{12}}) = [\mathbf{F}_s,\mathbf{F_n}]\,\tilde{\mathbf{D}}\,[\mathbf{F}_s,\mathbf{F_n}]^H \quad (14)$$

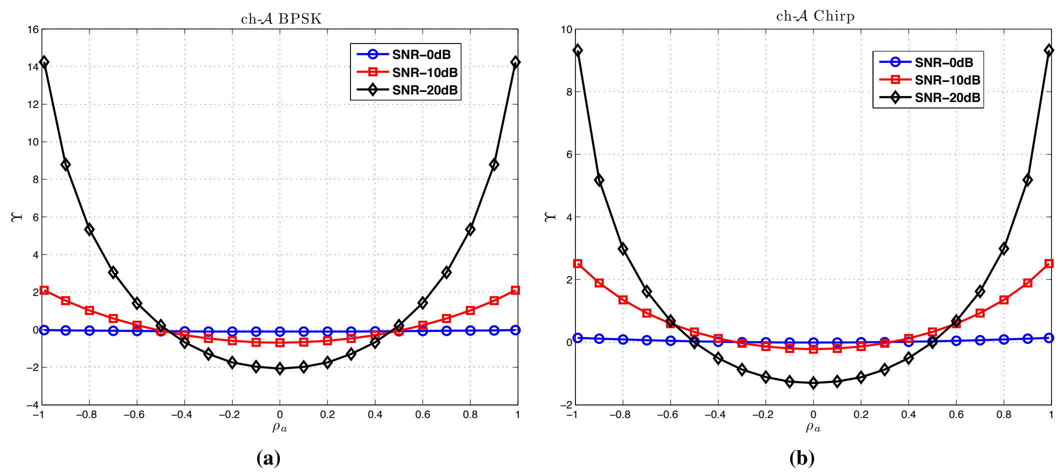

Fig. 1: For ch-$\mathcal{A}$ metric Υ vs ρ_a for SNR={0,10,20}dBs (a) BPSK, $N = 10$, (b) Chirp waveform, $N = 250$

varying channels, we have compared the information gain for TR and conventional, Gaussian channel models. Analytical and numerical evaluations demonstrated that TR may still be beneficial in channels which are not necessarily identical but still correlated over time. How much TR outperforms a conventional channel (if at all) depends on the waveform transmitted, the SNR and the degree of correlation present in the channel.

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