Problem 1. Textbook problem 6.1.3. Do it on your own rather than looking at the solution.

Solution 1:

a- The fact that the first correct answer needs to be on the \textit{nth} phone call means, there were \( n - 1 \) wrong calls before that

\[
P_{N_1}[n] = \begin{cases} 
\left( \frac{3}{4} \right)^{n-1} \frac{1}{4} & n = 1, 2, \ldots \\
0 & \text{otherwise}
\end{cases}
\]

b- \( N_1 \) is geometric with parameter \( p = \frac{1}{4} \). For a geometric RV then mean is \( E(N_1) = \frac{1}{p} = 4 \).

c- The fact that the fourth is call is the fourth correct answer needs to be on the \textit{nth} phone call means that there were \( n - 1 \) calls contains three correct answers

\[
P_{N_1}[n] = \begin{cases} 
\left( \frac{3}{4} \right)^{n-1} \left( \frac{3}{4} \right)^{n-4} \left( \frac{1}{4} \right)^4 & n = 1, 2, \ldots \\
0 & \text{otherwise}
\end{cases}
\]

d- By summing up the means of 4 identically distributed geometric random variables each with mean 4, we get \( E[N_4] = 4E[N_1] = 16 \).

Problem 2. Let \( U \) be a continuous random variable with uniform distribution over \([u_1, u_2]\).

- Find the moment generating function of \( U \).
- Use the MGF to calculate the first moment of \( U \).
- Use the MGF to calculate the second moment of \( U \).

Solution 2: From the definition of the characteristic function

\[
\phi_U(s) = E(e^{su}) = \int_{u_1}^{u_2} e^{su} \frac{1}{u_2 - u_1} du = \frac{e^{su_2} - e^{su_1}}{s(u_2 - u_1)}
\]
The first moment is

\[ E[U] = \frac{d\phi_U(s)}{ds} \bigg|_{s=0} = s \left[ u_2 e^{u_2 s} - u_1 e^{u_1 s} \right] - \left[ e^{u_2 s} - e^{u_1 s} \right] \bigg|_{s=0}. \] (2)

However evaluating this at \( s = 0 \) gives \( \frac{0}{0} \). So instead we have to use the Hopital’s rule

\[ E[U] = \lim_{s \to 0} \frac{u_2 e^{u_2 s} - u_1 e^{u_1 s} + s\left[ u_2^2 e^{u_2 s} - u_1^2 e^{u_1 s} \right] - \left[ u_2 e^{u_2 s} - u_1 e^{u_1 s} \right]}{2(u_2 - u_1)} \] (3)

\[ = \frac{u_1 + u_2}{2}. \] (4)

To find the second moment we first find the second derivative of \( \phi_U(s) \)

\[ E[U^2] = \frac{d^2\phi_U(s)}{ds^2} = \frac{s^2\left[ u_2^2 e^{u_2 s} - a^2 e^{u_1 s} \right] - 2s\left[ u_2 e^{u_2 s} - u_1 e^{u_1 s} \right] + 2\left[ u_2 e^{u_2 s} - u_1 e^{u_1 s} \right]}{(b-a)^3s^3} \] (5)

\[ = \frac{u_2^3 - u_1^3}{3(u_2 - u_1)} = \frac{(u_2^2 + u_1u_2 + u_1^2)}{3}. \] (6)

Once again use the Hopital’s Rule.

\[ E[U^2] = \lim_{s \to 0} \frac{d^2\phi_U(s)}{ds^2} = \lim_{s \to 0} \frac{s^2\left[ u_2^2 e^{u_2 s} - u_1^2 e^{u_1 s} \right]}{3(u_2 - u_1)} = \frac{u_2^3 - u_1^3}{3(u_2 - u_1)} = \frac{(u_2^2 + u_1u_2 + u_1^2)}{3}. \] (7)

**Problem 3.** Textbook problem 6.7.3 includes Matlab. Do it on your own rather than looking at the solution.

**Solution 3:**

a- The number of tests needed to identify 500 acceptable circuits is \( \text{Pascal}(k = 500, p = 0.8) \), which has \( E(L) = \frac{k}{p} = 625 \) tests.

b- Let \( k \) denote the number of acceptable circuits in \( n=600 \) tests. Since \( k \) is binomial (\( n=600,p=0.8 \))

\[ E[k] = np = 480 \] (8)

\[ \text{Var}[k] = np(1-p) = 96. \] (9)

Using the central limit theorem we estimate

\[ P(K \geq 500) = P\left( K - \frac{500}{\sqrt{96}} \geq \frac{20}{\sqrt{96}} \right) = Q\left( \frac{20}{\sqrt{96}} \right) = 0.026 \] (10)
c- Using MATLAB and the command

\[ 1 - \text{binomialcdf}(600,0.8,499) \] (11)

we obtain 0.025.

d- We need to find the smallest value of \( n \) such that the binomial Random Variable \( K \) satisfies \( P[k \geq 500] \geq 0.9 \). Since \( E[K] = np \) and \( \text{Var}[k] = np(1 - p) \), the CLT is

\[ P[K \geq 500] = P\left[ \frac{K - np}{\sqrt{np(1 - p)}} \geq \frac{500 - np}{\sqrt{np(1 - p)}} \right] = 1 - \phi(z) = 0.9 = \phi(-z). \] (12)

Then \( z = -1.29 \). Since \( p = 0.8 \), we have that

\[ np - 500 = 1.29 \sqrt{np(1 - p) \] (13)

\[ n = 641.3 \] (14)

**Problem 4.** Use the MGF to find the 1st, 2nd, 3rd and 4th moments of a Gaussian random variable with mean 0 and variance \( \sigma^2 \).

**Solution 4:** Using the moment generating function of \( X \),

\[ \phi_X(s) = e^{\frac{s^2 \sigma^2}{2}}. \] (15)

With this we can get the \( n \)th moments by taking the \( n \)th derivative with respect to \( s \) and setting \( s = 0 \).

\[ E(X) = \sigma^2 s e^{\frac{s^2 \sigma^2}{2}} \bigg|_{s=0} = 0 \] (16)

\[ E(X^2) = \sigma^2 e^{\frac{s^2 \sigma^2}{2}} + \sigma^4 s^2 e^{\frac{s^2 \sigma^2}{2}} \bigg|_{s=0} = \sigma^2 \] (17)

\[ E(X^3) = (3\sigma^4 s + \sigma^6 s^3)e^{\frac{s^2 \sigma^2}{2}} \bigg|_{s=0} = 0 \] (18)

\[ E(X^4) = (3\sigma^4 + 6\sigma^6 s^2 + \sigma^8 s^4)e^{\frac{s^2 \sigma^2}{2}} \bigg|_{s=0} \] (19)

As for \( Y = X + \mu \) so that \( Y \) is \( \mathcal{N}(\mu, \sigma) \)

\[ E(Y^2) = E[(X + \mu)^2] = E[X^2 + \mu^2 + 2\mu X] = \sigma^2 + \mu^2. \] (20)

\[ E(Y^3) = E[(X + \mu)^3] = E[X^3 + \mu^3 + 3\mu X^2 + 3X \mu^2] = 3\mu \sigma^2 + \mu^3 \] (21)

\[ E(Y^4) = E[(X + \mu)^4] = E[X^4 + \mu^4 + 4\mu X^3 + 6\mu^2 X^2 + 4\mu^3 X] = 3\sigma^4 + 6\mu^2 \sigma^2 + \mu^4 \] (22)

**Problem 5.** Textbook problem 6.6.2. Do it on your own rather than looking at the solution.

**Solution 5:** Knowing that the probability that the voice calls occurs is 0.8 and the probability that the data call occurs is 0.2 we define the random \( D_i \) as the number of data calls in a single
telephone call. It is obvious that for any \( i \) there are only two possible values for \( D_i \), namely 0 and 1. Furthermore for all \( i \) the \( D_i \)s are independent and identically distributed with the following pmf.

\[
P_D[d] = \begin{pmatrix}
0.8 & d = 0 \\
0.2 & d = 1
\end{pmatrix}
\]

Then

\[
E[D] = 0.2 \\
Var[D] = 0.16.
\]

and then

\begin{enumerate}
\item \( E(K_{100}) = 100E[D] \)
\item \( Var[K_{100}] = \sqrt{100Var[D]} = 4 \)
\item \( P(K_{100} \geq 18) = 1 - \Phi\left(\frac{18-20}{4}\right) = 1 - \Phi(-0.5) = \Phi(0.5) = 0.6915. \)
\item \( P[16 \leq k_{100} \leq 24] = 1 - \Phi\left(\frac{24-20}{4}\right) = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = 0.6826. \)
\end{enumerate}