Problem 1. A certain deep space transmitter uses on-off modulation of a laser to send a bit, with value either zero or one. If the bit is zero, the number of photons, $X$, arriving at the receiver has the Poisson distribution with mean $\lambda_0 = 2$; and if the bits is one, $X$ has the Poisson distribution with mean $\lambda_1 = 6$. A decision rule is needed to decide, based on observation of $X$, whether the bit was a zero or a one.

1. Suppose we decide that a 1 is sent if $P[X = k | \text{one is sent}] > P[X = k | \text{zero is sent}]$. What does this decision rule simplify to in terms of $X$?

2. Suppose we know that the probability of sending a zero is $\pi_0$ and that of sending a one is $\pi_1$. Suppose now that we decide a 1 is sent if $P[X = k, \text{one is sent}] > P[X = k, \text{zero is sent}]$. What does this decision rule simplify to now in terms of $X$ if $\pi_0 = 5\pi_1$?

Solution 1:

1. Decision Rule is in favor of 1 if

$$P[X = k | \text{one is sent}] \geq P[X = k | \text{zero is sent}] \quad (1)$$

Thus

$$\frac{P[X = k | \text{one is sent}]}{P[X = k | \text{zero is sent}]} \geq 1 \quad (3)$$

But it is given that the number of photons $X$ arriving at the receiver has Poisson distribution, then

$$\frac{P[X = k | \text{one is sent}]}{P[X = k | \text{zero is sent}]} = \frac{e^{-\lambda_1} \frac{\lambda_1^k}{k!}}{e^{-\lambda_0} \frac{\lambda_0^k}{k!}} = \left(\frac{\lambda_1}{\lambda_0}\right)^k e^{-(\lambda_1-\lambda_0)} = 3^k e^{-4} \approx \frac{3^k}{54.6} \quad (4)$$

Thus based on this result, we decide that a bit 1 is sent if $\frac{3^k}{54.6} \geq 1$, that is if $X = k \geq 4$. 
2. Now Decision Rule is in favor of 0 if

\[ P[X = k, \text{one is sent}] \geq P[X = k, \text{zero is sent}] \]  \hspace{1cm} (5)

\[ P[X = k|\text{one is sent}]P[\text{one is sent}] \geq P[X = k|\text{zero is sent}]P[\text{zero is sent}] \]  \hspace{1cm} (6)

\[ P[X = k|\text{one is sent}] \pi_1 \geq P[X = k|\text{zero is sent}] \pi_0 \]  \hspace{1cm} (7)

\[ \frac{P[X = k|\text{one is sent}]}{P[X = k|\text{zero is sent}]} \geq \frac{\pi_0}{\pi_1} = 5 \]  \hspace{1cm} (8)

\[ \frac{P[X = k|\text{one is sent}]}{P[X = k|\text{zero is sent}]} = \frac{3^k}{54.6} \geq 5 \]  \hspace{1cm} (9)

Thus the condition above is satisfied when \( X \geq 6 \)

**Problem 2.** Let \( X \) have CDF shown in Fig. 3.3. Find the numerical values of the following quantities:

1. \( P[X \leq 1] \)
2. \( P[X \leq 10] \)
3. \( P[X \geq 10] \)
4. \( P[X = 10] \)
5. \( P[|X - 5| \leq 0.1] \)

![CDF Graph](image_url)

**Solution 2:**

- \( P[X \leq 1] = F_X[1] = 0.05 \) where \( F_X(1) \) is calculated from the equation of the line \( F_X[x] = 0.05x \) when \( x < 5 \)
- \( P[X \leq 10] = F_X[10] = 0.75 \) from the graph.
- \( P[X \geq 10] = 1 - P(X < 10) = 1 - F_X(10^-) = 0.5. \)
• $P[X = 10] = F_X(10^+) - F_X(10^-) = 0.75 - 0.5 = 0.25$.
• $P[|X - 5| \leq 0.1] = P[4.9 \leq X \leq 5.1] = P(X \leq 5.1) - P(X < 4.9) = 0.5 - 0.245 = 0.255$.

where $F_X(4.9)$ is calculated from the equation of the line $F_X[x] = 0.05x$ when $x < 5$

**Problem 3.** Exponential random variables have a nice property termed the “memoryless property”, which means that $P[T > s + t | T > s] = P[T > t]$, where $T$ is exponentially distributed. Show this property.

**Solution 3:** The aim is to prove that exponential random variables have property of being memoryless i.e $P[T > s + t | T > s] = P[T > t]$.

Using Bayes Rule

\[
P[T > s + t | T > s] = \frac{P[T > s + t, T > s]}{P[T > s]} \tag{14}
\]
\[
= \frac{P[T > s + t | T > s]}{P[T > s]}
\tag{15}
\]
\[
= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \tag{16}
\]
\[
= e^{-\lambda t} = P[T > t] \tag{17}
\]

**Problem 4.** Suppose $Y = X^2$, where $X \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = 2, \sigma^2 = 3$. Find the pdf of $Y$.

*(HINT: $\frac{d}{dx} \Phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$).*

**Solution 4:** The RV $Y = g(X)$. The RV $X$ can take on all real values and $Y$ take on only values over $\mathbb{R}^+$

- if $y < 0$ then $F_Y(y) = 0$ since $P(Y \geq 0) = 1$
- if $y \geq 0$ then
\[ F_Y(y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) \]
\[ = P\left(\frac{-\sqrt{y} - 2}{\sqrt{3}} \leq \frac{X - 2}{\sqrt{3}} \leq \frac{\sqrt{y} - 2}{\sqrt{3}}\right) = \Phi\left(\frac{\sqrt{y} - 2}{\sqrt{3}}\right) - \Phi\left(\frac{-\sqrt{y} - 2}{\sqrt{3}}\right) \] (18)

Taking the derivative with respect to \( y \), using the chain rule and the hint that \( \frac{d}{dx} \Phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \)

\[ f_Y(y) = \begin{cases} 
\frac{1}{\sqrt{2\pi y}} \left[ \exp\left(-\frac{(\sqrt{y} - 2)^2}{6}\right) + \exp\left(-\frac{(\sqrt{y} + 2)^2}{6}\right) \right] & y \geq 0 \\
0 & y < 0 
\end{cases} \]