Problem 1. Given a uniform, continuous random variable $X$ whose range is $[-3, 3]$. We quantize $X$ to give $Y$, using $L$ levels such that SNR$_Q$(dB) = 25. Find:

1. $E[X^2]$
2. $Var[X]$
3. $\mu_X$
4. $E[(X - Y)^2]$
5. $L$
6. Verify using Matlab that your choice of $L$ yields SNR$_Q$(dB) $\approx$ 25

Solution 1:
The PDF of $X$ is a continuous Random Variable then
$$f_X(x) = \begin{cases} \frac{1}{6} & -3 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

The Variance for a Uniform Random Variable is as follows
$$Var[X] = \frac{(b-a)^2}{12}$$

1. $E(X^2)$ is calculated as follows
$$E(X^2) = \int_{-3}^{3} f_X(x)x^2 dx = \int_{-3}^{3} \frac{1}{6}x^2 dx = 3$$
2. $Var[X]$ is calculated as follows
$$Var[X] = \frac{(b-a)^2}{12} = \frac{(3-(-3))^2}{12} = 3$$
3. $\mu_X$ is calculated as follows
$$\mu_X = \sqrt{E(X^2)} - Var(X) = \sqrt{3 - \frac{3}{3}} = 0$$
4. Number of levels $L$ is calculated as follows
\[
10 \log SNR = 25 \Rightarrow SNR = 10^{2.5} = 316.23 \tag{5}
\]
\[
L^2 = 316.23 \Rightarrow L = 17.778 \tag{6}
\]
But $L$ can only be integer representing number of levels for quantization thus choose $L = 18$.

5. $E(X - Y)^2$ known as the mean square error. Where $(X-Y)$ represent the error due to quantizations is calculated as follows
\[
\Delta = \frac{b - a}{L} = \frac{6}{18} = \frac{1}{3} \tag{7}
\]
\[
E(X - Y)^2 = \frac{\Delta^2}{12} = \frac{1^2}{3} = 0.00926. \tag{8}
\]

6. MATLAB code is as follows:

```matlab
>> X = rand(1,1e7)*6-3;
>> delta = 6/L;
>> Y = round(X/delta)*delta;
>> Z = X-Y;
>> mean(Z.^2)
ans =
    0.00925335563324
>> 10*log10(mean(X.^2)/mean(Z.^2))
ans =
     25.10618609681968
```

**Problem 2.** Let $W = \text{Bernoulli}(1/2)$ and $X = 10W - 5$ and $Y = X + N$, where $N$ is a Gaussian random variable having zero mean. Define the SNR as $E[X^2]/E[N^2]$, or in decibels, \(\text{SNR(dB)} = 10 \log_{10}(E[X^2]/E[N^2])\). Define the decoder output (our decision on which bit was transmitted based on the received signal $Y$) as a new random variable $Z$ which is equal to 0 is $Y < 0$ and equal to 1 is $Y \geq 0$. If $Z = W$ then no error occurs, if $Z \neq W$ then an error occurred due to the additive Gaussian noise in the channel. Find:

1. The variance of $N$ when the channel SNR is 30dB.
2. The channel SNR in dB when the variance of the RV \( N \) is 0.1.

3. The power of the noise in the channel is \( E[N^2] \). Derive an expression for the probability of a bit error in the channel (probability that \( Z \neq W \)) in terms of the noise power.

4. Derive an expression for the probability of a bit error in the channel in terms of channel SNR.

Solution 2: \( W = \text{Bernoulli} (1/2) \). Thus \( W \) takes on values either 1 or 0 with equal probability \( p = 0.5 \) which makes the range of values for \( X = 5 \) or -5 since \( X = 10W - 5 \). It is important for this problem to calculate the \( E(X^2) \) which is:

\[
E(X^2) = (5)^2(0.5) + (-5)^2(0.5) = 25
\]

1. The variance of \( N \) can be calculated from the SNR.

\[
10 \log \text{SNR} = 10 \log \frac{E(X^2)}{E(N^2)} = 30
\]

\[
\frac{E(X^2)}{E(N^2)} = 1000 \quad \text{with} \quad E(X^2) = 25
\]

\[
E(N^2) = \frac{25}{1000} = \frac{1}{40}
\]

Note that \( E(N^2) = \sigma_N^2 \) because \( N \) is zero mean thus \( E(N) = 0 \).

2. SNR is calculated as follows

\[
\text{SNR}(dB) = 10 \log \frac{E(X^2)}{\sigma_N^2} = 10 \log \frac{25}{0.1} = 23.9794
\]

3. The Probability of a bit error in the channel (probability that \( Z \neq W \))

\[
P(\text{bit error}) = P(Z \neq 1|W = 1)P(W = 1) + P(Z \neq 0|W = 0)P(W = 0)
\]

\[
= 0.5P(Y < 0|W = 1) + 0.5P(Y \geq 0|W = 0)
\]

\[
= 0.5P(X + N < 0|W = 1) + 0.5P(X + N \geq 0|W = 0)
\]

\[
= 0.5P(N < -5) + 0.5P(N \geq 5)
\]

But the the above probabilities are equal then

\[
P(N \geq 5) = Q\left(\frac{5}{E[N^2]}\right)
\]

4. The expression for the probability of a bit error in the channel in terms of channel SNR, from the above part it can be seen that:

\[
P(\text{bit error}) = Q\left(\sqrt{\frac{E[X^2]}{E[N^2]}}\right) = Q(\sqrt{SNR})
\]
Problem 3. Let \((X, Y)\) have the joint pmf given in the table below.

\[
\begin{array}{c|ccc}
Y = 3 & 0.1 & 0.1 & 0 \\
Y = 2 & 0 & 0.2 & 0.2 \\
Y = 1 & 0 & 0.3 & 0.1 \\
\hline
X = 1 & \quad & \quad & \quad \\
X = 2 & \quad & \quad & \quad \\
X = 3 & \quad & \quad & \quad \\
\end{array}
\]

Find:

1. The pmf of \(X\)
2. The pmf of \(Y\)
3. \(P[X = Y]\)
4. \(P[X > Y]\)

Solution 3:
(a) The pmf of \(X\) is given by the column sums:

\[
P_X(1) = 0.1, P_X(2) = 0.3 + 0.2 + 0.1 = 0.6, P_X(3) = 0.1 + 0.2 = 0.3.
\]

(b) The pmf of \(Y\) is given by the rows sums:

\[
P_Y(1) = 0.3 + 0.1 = 0.4, P_Y(2) = 0.2 + 0.2 = 0.4, P_Y(3) = 0.1 + 0.1 = 0.2.
\]

(c) For \(P[X = Y]\)

\[
P(X = Y) = p_{X,Y}(1,1) + p_{X,Y}(2,2) + p_{X,Y}(3,3) = 0 + 0.2 + 0 = 0.2.
\]

(d) For \(P[X > Y]\)

\[
P(X > Y) = p_{X,Y}(2,1) + p_{X,Y}(3,1) + p_{X,Y}(3,2) = 0.3 + 0.1 + 0.2 = 0.6.
\]