Problem 1. Textbook problem 5.4.5. Do it on your own rather than looking at the solution.

Solution 1: Evaluate the joint density function, it is not equal to the product of marginal pdfs then they are not independent

\[ f_{X_1,X_2,X_3}(10,9,8) = 0 \neq f_{X_1}(10)f_{X_2}(9)f_{X_3}(8) \quad (1) \]

Problem 2. Textbook problem 5.6.1. Do it on your own rather than looking at the solution.

Solution 2:

\[ C_X = \begin{bmatrix} \text{Var}(X_1) & \text{Cov}(X_1, X_2) \\ \text{Cov}(X_1, X_2) & \text{Var}(X_2) \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix} \quad (2) \]

b- Let \( Y = [X_1 \ X_2]' \) since \( Y \) is linear combination of gaussian random variables then it is also a gaussian vector.

\[ C_Y = A C_X A^T = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 28 & -66 \\ -66 & 252 \end{bmatrix} \quad (3) \]

Problem 3. Textbook problem 5.7.1. Do it on your own rather than looking at the solution.

Solution 3: The correlation matrix is

\[ R_x = C_x + \mu_x\mu_x' = \begin{bmatrix} 20 & 30 & 25 \\ 30 & 68 & 46 \\ 25 & 46 & 40 \end{bmatrix} \quad (4) \]

b- Let \( Y = [X_1 \ X_2]' \) since \( Y \) is subset of gaussian then it is also a gaussian vector

\[ E(Y) = [E(X_1) \ E(X_2)]' = [4 \ 8]' \quad (5) \]

\[ C_Y = \begin{bmatrix} \text{Var}(X_1) & \text{Cov}(X_1, X_2) \\ \text{Cov}(X_1, X_2) & \text{Var}(X_2) \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \quad (6) \]

\[ C_Y^{-1} = \frac{1}{20} \begin{bmatrix} 4 & +2 \\ +2 & 4 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \quad (7) \]
Now solving for the exponent \((Y - \mu_Y)K_Y^{-1}(Y - \mu_Y)\)

\[
\begin{bmatrix}
    y_1 - 4 & y_2 - 8
\end{bmatrix}
\begin{bmatrix}
    \frac{1}{3} & \frac{1}{3} \\
    \frac{1}{3} & \frac{1}{3}
\end{bmatrix}
\begin{bmatrix}
    y_1 - 4 \\
    y_2 - 8
\end{bmatrix}
= \frac{y_1^2}{3} + \frac{y_1y_2}{3} - \frac{16y_2}{3} - \frac{20y_2}{3} + \frac{y_2^2}{3} + \frac{112}{3}
\] (10)

The PDF of \(Y\) is

\[
f_Y(y) = \frac{1}{\sqrt{48\pi^2}} e^{-\frac{y_1^2}{3} - \frac{y_1y_2}{3} - \frac{16y_2}{3} - \frac{20y_2}{3} + \frac{y_2^2}{3} + \frac{112}{3}}
\] (11)

c- With the use of \(Q\)-function

\[
P(X_1 > 8) = Q(2) = 0.0288
\] (12)

**Problem 4.** Suppose \(X\) and \(Y\) are jointly Gaussian distributed with parameters \(E[X] = \mu_x\), \(E[Y] = \mu_y\), \(Var(X) = \sigma_x^2\), \(Var(Y) = \sigma_y^2\) and \(E[XY] = \rho_{XY}\). Are the following statements true or false?

1. \(X\) and \(Y\) are then also Gaussian with \(X \sim \mathcal{N}(\mu_x, \sigma_x^2)\), and \(Y \sim \mathcal{N}(\mu_y, \sigma_y^2)\).
2. If \(\rho_{XY} = 0\) then \(X\) and \(Y\) are independent.
3. \(Z = aX + bY\) for known constants \(a, b\) is Gaussian with mean \(a\mu_x + b\mu_y\).
4. \(Z = aX + bY\) for known constants \(a, b\) is Gaussian with variance \(a^2\sigma_x^2 + b^2\sigma_y^2\).
5. The linear minimum mean squared estimator of \(X\) in terms of \(Y\) is \(\hat{X} = \mu_x + \rho_{XY}(Y - \mu_y)\frac{\sigma_x}{\sigma_y}\).
6. The conditional distribution of \(Y|X\) is Gaussian.
7. \(E[X|Y] = \mu_x + \rho_{XY}(Y - \mu_y)\frac{\sigma_x}{\sigma_y}\).

**Solution 4:**

1. \(X\) and \(Y\) are then also Gaussian with \(X \sim \mathcal{N}(\mu_x, \sigma_x^2)\), and \(Y \sim \mathcal{N}(\mu_y, \sigma_y^2)\).
   Yes, this is true. A joint Gaussian distribution implies marginally distributed gaussian.

2. If \(\rho_{XY} = 0\) then \(X\) and \(Y\) are independent.
   \(\rho_{XY} = 0\) then \(Cov(X,Y) = 0\) and \(E(XY) = E(X)E(Y)\) then \(X\) and \(Y\) are uncorrelated and in the case of gaussian only uncorrelated random variables implies independent.

3. \(Z = aX + bY\) for known constants \(a, b\) is Gaussian with mean \(a\mu_x + b\mu_y\).
   This is true, \(E(Z) = aE(X) + bE(Y) = a\mu_x + b\mu_y\)

4. \(Z = aX + bY\) for known constants \(a, b\) is Gaussian with variance \(a^2\sigma_x^2 + b^2\sigma_y^2\).
   This is not true, since we did not account for the covariance between \(X\) and \(Y\).
   \(\text{Var}(Z) = 2ab\text{Cov}(X,Y) + \text{Var}(X) + \text{Var}(Y)\)
5. The linear minimum mean squared estimator of $X$ in terms of $Y$ is $\hat{X} = \mu_X + \rho_{XY}(Y - \mu_Y)\frac{\sigma_X}{\sigma_Y}$. True. This is the optimal estimator.

6. The conditional distribution of $Y|X$ is Gaussian.
   True, Conditioning preserves Gaussianity.

7. $E[X|Y] = \mu_X + \rho_{XY}(Y - \mu_Y)\frac{\sigma_X}{\sigma_Y}$. True in fact this is the Linear minimum mean square estimate of $X$.

**Problem 5.** Suppose $X$ and $Y$ are zero-mean unit-variance jointly Gaussian random variables with correlation coefficient $\rho = 0.5$.

1. Find $\text{Var}(3X - 2Y)$.
2. Find the numerical value of $P[(3X - 2Y)^2 \leq 28]$.
3. Find the numerical value of $E[Y|X = 3]$.

**Solution 5:**

\[
\text{Var}(3X - 2Y) = \text{Cov}(3X - 2Y, 3X - 2Y) = 9\text{Var}(X) + 4\text{Var}(Y) - 12\text{Cov}(XY) = 9 - 6 + 4 = 7
\]

Find the numerical value of $P[(3X - 2Y)^2 \leq 28]$.
Define a Random variable $Z = 3X - 2Y$. Keeping in mind that any linear combination of Gaussian is Gaussian then $Z$ is a Gaussian with $E(Z) = 0$ and $\text{Var}(Z) = 7$

\[
P(Z^2 \leq 28) = P(-\sqrt{28} \leq Z \leq \sqrt{28}) = \Phi\left(\frac{\sqrt{28}}{7}\right) - \Phi\left(-\frac{\sqrt{28}}{7}\right) = 2(\Phi(2) - 0.5) = 0.9545.
\]

\[
E[Y|X = 3] = \mu_Y + (x - \mu_X)\frac{\text{Cov}(XY)}{\text{Var}(X)}
= 0 + (3)\rho_{X,Y} = 1.5
\]