Types of random processes:

→ Look at picture.

Expected values + Correlation of Random processes

For a random process $X(t)$, define: $t \in \mathbb{R}$

- Mean: $\mu_X(t) \triangleq E[X(t)]$ (like mean $\mu_X = E[X]$) regular ~~and~~ random variable

  → deterministic function of time! ensemble average.

- Autocorrelation: $R_X(t,\tau) \triangleq E[X(t)X(t+\tau)]$

  (like correlation btw. X and Y is defined as $E[XY]$)

- Autocovariance: $C_X(t,\tau) \triangleq Cov(X(t), X(t+\tau))$

  (like covariance btw. X and Y is $Cov(X,Y) = E[(X-E[X])(Y-E[Y])]$)

Likewise, for a random sequence X_n, define: $n \in \mathbb{Z}$

- Mean: $\mu_X[n] = E[X_n]$ is a deterministic function of n!
- Autocorrelation: $R_X[m,k] = E[X_m X_{m+k}]$ $n, k \in \mathbb{Z}$
- Auto covariance: $C_X[m,k] = Cov[X_m, X_{m+k}]$

I.I.D. sequence : (independent, identically distributed sequence)

This is a random sequence X_n $(\cdots \cancel{X}_{-2}, X_{-1}, X_0, X_1, X_2, X_3, \cdots)$ where all X_i are independent and identically distributed random variables, ie.

$$\forall n \quad P_{X_n}(x_n) = P_X(x) \quad \text{for some } \overset{\text{discrete}}{\wedge} \text{ RV } X$$

$$\text{OR } \forall n \quad f_{X_n}(x_n) = f_X(x) \quad \text{for some continuous RV } X$$

E.g: Let X_T be a randon sequence for $T = 0, 1, 2, \cdots$ which is equal to the # dots facing up on a dice roll at time T.

→ This is an IID sequence! Let X = dice roll

$$P_X(x) = \begin{cases} 1/6 & x = 1, 2, 3, \ldots, 6 \\ 0 & \text{else.} \end{cases}$$

$$\rightarrow \mu_X[T] = \mu_X$$

$$= \frac{1}{6}\left[1+2+3+4+5+6\right] = 3.5$$

For IID sequences in general, we have:

- $\mu_X[n] = \mu_X$ (constant mean! NOT a function of n)
- $\text{Var}[X_n] = \sigma_X^2$
- $R_X[m,k] = E[X_m X_{m+k}]$

$$= \begin{cases} E[X_m^2] = \sigma_X^2 + \mu_X^2 & \text{if } k = 0 \\ E[X_m]\,E[X_{m+k}] = \mu_X^2 & \text{if } k \neq 0 \end{cases}$$

EX: Input to a digital filter is a iid random sequence

$\cdots X_{-1}, X_0, X_1, \cdots$ with $E[X_i] = 0$, $Var[X_i] = 1$ $\forall i$.

$\cdots X_{-1}, X_0, X_1, \cdots \longrightarrow$ Digital filter $\longrightarrow$ The output $\cdots Y_{-1}, Y_0, Y_1, \cdots$ which is expressed as:

$$Y_n = X_n + X_{n-1} \quad \forall \text{ integers } n.$$

- Find $E[Y_n]$: $E[Y_n] = E[X_n + X_{n-1}]$

$$= E[X_n] + E[X_{n-1}]$$
$$= 0 + 0$$
$$= 0 \qquad \forall n.$$

- Find the autocovariance function $C_Y[m,k]$:

$$C_Y[m,k] \triangleq E\left[\left(Y_m - \underbrace{E[Y_m]}_{0}\right)\left(Y_{m+k} - \underbrace{E[Y_{m+k}]}_{0}\right)\right]$$

$$= E[Y_m Y_{m+k}] \qquad \text{as } E[Y_m] = 0 \ \forall m.$$

$$= E[(X_m + X_{m-1})(X_{m+k} + X_{m+k-1})]$$

$$= E[X_m X_{m+k}] + E[X_m X_{m+k-1}] + E[X_{m-1} X_{m+k}] + E[X_{m-1} X_{m+k-1}].$$

$$= C_X[m,k] + C_X[m,k-1] + C_X[m-1,k+1] + C_X[m-1,k].$$

$\rightarrow$ Need to get $C_X[m,k]$!!

Notice that since $E[X_n]=0$ $\forall n$, and X_n is iid.,

$$C_X[m,k] = R_X[m,k]$$
$$= \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{else.} \end{cases}$$

$$C_X[m,k] \triangleq E\big[(X_m - \underbrace{E[X_m]}_{0})(X_{m+k} - \underbrace{E[X_{m+k}]}_{0})\big]$$
$$= E[(X_m)(X_{m+k})] \quad \text{since } E[X_m]=0 \ \forall m.$$

So,
$$C_Y[m,k] = C_X[m,k] + C_X[m,k-1] + C_X[m-1,k+1] + C_X[m-1,k].$$

Cases: $k=0$: $C_Y[m,0] = C_X[m,0] + C_X[m,-1] + C_X[m-1,1] + C_X[m-1,0]$
$$= 1+0+0+1 = 2.$$

$k=-1$: $C_Y[m,-1] = C_X[m,-1] + C_X[m,-2] + C_X[m-1,0] + C_X[m-1,-1]$
$$= 1$$

$k=+1$: $C_Y[m,1] = 1$

Thus,
$$C_Y[m,k] = \begin{cases} 2-|k| & k=-1,0,1 \\ 0 & \text{else.} \end{cases}$$
interpret!