We've seen IID sequences; these are a special class of random processes.

Stationary processes

In general if $X(t)$ is a random process $\Rightarrow \forall t_i$, $X(t_i)$ is a RV

~~Usually~~ In general, $f_{X(t_1)}(x)$, the pdf of $X(t_1)$ depends on t_1.

For stationary processes it does not, i.e.

$$f_{X(t_1)}(x) = f_{X(t_1+\tau)}(x) = f_X(x) \quad \forall \tau, t_1 \in \mathbb{R}.$$

"1st order stationary"

"m-th order stationary": means

$$f_{(X(t_1)\cdots X(t_m))}(x_1,\dots,x_m) = f_{(X(t_1+\tau)\cdots X(t_m+\tau))}(x_1,\dots,x_m) \quad \forall \tau, t_1,\dots, t_m.$$

A random process $X(t)$ is called "strictly stationary" if it is m-th order stationary for $m=1,2,3,\dots$ (all m!!)

If $X(t)$ is strictly stationary then,

- $\mu_X(t) = \mu$ (mean is constant). easy way to check whether $X(t)$ could be stationary.
- $R_X(t,\tau) \triangleq E[X(t)X(t+\tau)] = R_X(0,\tau) = R_X(\tau)$

(correlation function only a function of time difference τ)

$X(t)$ is "wide-sense-stationary" if (practically relevant, used everywhere)

- $\mu_X(t) = \mu_X$ (mean is constant) (WSS)
- $R_X(t,\tau) = R_X(\tau)$ (~~covariance~~ correlation function a function of τ only!)

$\Rightarrow$ no m-th order stationary conditions are required!

X_n is a "wide-sense stationary" random sequence if:

- $E[X_n] = \mu_X$ (constant mean, not a function of n)
- $R_X[m,k] = R_X[0,k] = R_X[k]$ (correlation sequence depends only on time difference k)

Strictly stationary $\Rightarrow$ WSS

strictly stationary $\not\Leftarrow$ WSS

Here's an example that shows WSS $\not\Rightarrow$ strictly stationary:

Consider a "non-iid" random sequence X_n where:

independent $\left\{ \begin{array}{l} X_{(2k)} \text{ is } \mathcal{N}(0, 1) \\ X_{(2k+1)} \text{ is uniform with mean 0 and variance 1} \end{array} \right\} k \in \mathbb{Z}$

What is $E[X_n]$? $E[X_n] = 0$ since both have mean 0.

What is $R_X[n,k]$?

$$R_X[n,k] = E[X_n X_{n+k}] = \begin{cases} E[X_m^2] & \text{if } k=0 \\ \underbrace{E[X_m]E[X_{m+k}]}_{0} & \text{if } k \neq 0. \end{cases}$$

$$= \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{else.} \end{cases}$$

Is X_n WSS? YES! 2 conditions are met.

Is X_n strictly stationary? NO! It's not even 1st order stationary as the pdf of $X_{2k} \neq$ pdf of X_{2k+1} so pdf changes over time.