EX: $X(t)$ is WSS, let $Y(t) = X(t)\cos(2\pi f_c t + \Theta)$, where $\Theta$ is uniform on $[0, 2\pi]$. $\Theta$ is independent of $X(t)$.

- what is the pdf of $\Theta$? $f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & \text{for } \theta \in [0, 2\pi] \\ 0 & \text{else.} \end{cases}$

- Is $Y(t)$ WSS?

$$\mu_Y(t) \overset{\text{DEF.}}{=} E[Y(t)] = E[X(t)\cos(2\pi f_c t + \Theta)]$$

$$= E[X(t)]\,E[\cos(2\pi f_c t + \Theta)]$$ as $X(t)$, $\Theta$ are independent.

$$= \mu_X \cdot \underbrace{\int_0^{2\pi} \frac{1}{2\pi} \cdot \cos(2\pi f_c t + \theta)\, d\theta}_{} = \frac{\mu_X}{2\pi}\Big[\underbrace{\sin(2\pi f_c t + 2\pi) - \sin(2\pi f_c t + 0)}_{0 \text{ as sin is periodic with period } 2\pi}\Big]$$

$$= \mu_X \cdot 0 = 0 \rightarrow \text{constant!}$$

$$R_Y(t, \tau) \overset{\text{DEF.}}{=} E[Y(t)Y(t+\tau)]$$

$$= E[X(t)\cos(2\pi f_c t + \Theta)\, X(t+\tau)\cos(2\pi f_c(t+\tau) + \Theta)]$$

independence of $\Theta, X(t)$

$$= \underbrace{E[X(t)X(t+\tau)]}_{}\, E[\cos(2\pi f_c t + \Theta)\cos(2\pi f_c(t+\tau) + \Theta)]$$

$X(t)$ is WSS. $\rightarrow$

$$= R_X(\tau)\, E\Big[\tfrac{1}{2}\big[\cos(2\pi f_c \tau) + \cos(2\pi f_c(2t+\tau) + 2\Theta)\big]\Big]$$ $\cos a \cos b = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$

$R_X(t, \tau)$

$$= \underbrace{R_X(\tau)\, \frac{1}{2} \cos(2\pi f_c \tau)}_{\text{is a function only of } \tau\,!!} + \underbrace{\frac{1}{2} E[\cos(2\pi f_c(2t+\tau) + 2\Theta)]}_{0}$$

since sin is periodic with period $2\pi$.

So, $Y(t)$ is WSS.

Properties of the autocorrelation function $R_X(\tau)$ of a WSS process $X(t)$:

- $R_X(\tau) = R_X(-\tau)$
- Signal power $\triangleq E[X^2(t)] = R_X(0)$ $\tau = 0$.

  DEF
- $|R_X(\tau)| \leq R_X(0)$.

Cross-correlation function

Let $X(t), Y(t)$ be two random process, then define

$$R_{XY}(t,\tau) \triangleq E[X(t)\,Y(t+\tau)].$$

Let $X_n, Y_n$ are two random sequences, define

$$R_{XY}[m,k] \triangleq E[X_m\,Y_{m+k}].$$

$X(t), Y(t)$ are jointly WSS if:

1) $X(t)$ is WSS
2) $Y(t)$ is WSS
3) $R_{XY}(t,\tau) = R_{XY}(\tau)$ (for continuous time RP)
4) $R_{XY}[n,k] = R_{XY}[k]$ (for discrete time RP).

# Power Spectral Density

Let $X(t)$ be a WSS random process with auto-correlation function $R_X(\tau)$.

Define the power spectral density (PSD) $\boxed{S_X(f)}$ of $X(t)$ as KNOW.

$$S_X(f) = \int_{-\infty}^{+\infty} R_X(\tau) e^{-j2\pi f\tau} \, d\tau. \quad \text{(Fourier transform of } R_X(\tau))$$

$$R_X(\tau) = \int_{-\infty}^{+\infty} S_X(f) e^{j2\pi f\tau} \, df$$

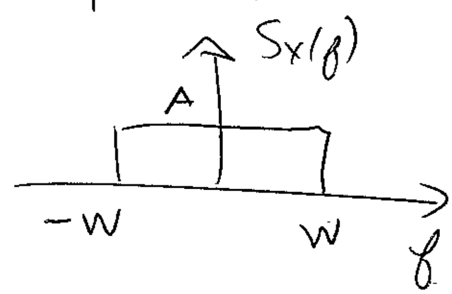
Fourier transform pair.

(You will be given Fourier transform tables if needed).

$$\underline{\text{Signal power}} \triangleq E[X^2(t)] = E[X(t)^2] = R_X(0) = \int_{-\infty}^{+\infty} S_X(f) \, df.$$

power spectral density!!

<u>EX</u>: Let $S_X(f) = \begin{cases} A & -W \le f \le W \\ 0 & \text{else} \end{cases}$




What is the signal's power?

$$R_X(0) = \int_{-\infty}^{+\infty} S_X(f) \, df = \int_{-W}^{W} A \, df = 2AW.$$

What is $R_X(\tau)$? $R_X(\tau)$ is the Fourier transform of $S_X(f)$.

$$R_X(\tau) = 2AW \frac{\sin 2\pi W\tau}{2\pi W\tau}$$

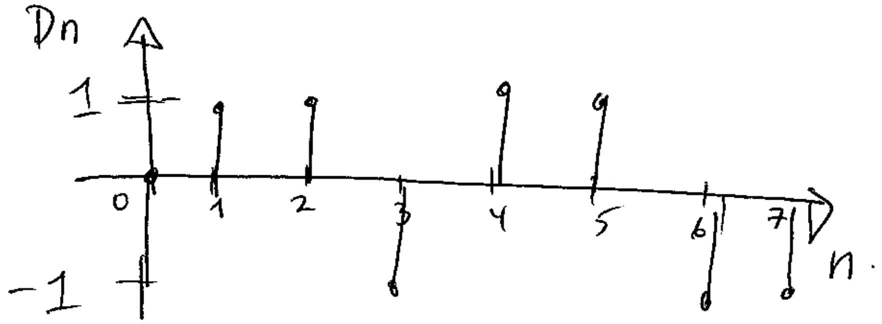
(sinc function!)

# EX: Random walk:

A particle moves along a line in time steps, ie. at time $n$ it either moves left 1 or ~~of~~ right 1 according to the outcome of a Bernoulli RV $I_n$ (at time $n$). Let $D_n$ denote the motion of the particle at time $n$, where $D_0 = 0$. Then $(n \geq 1)$.

$$D_n = \begin{cases} +1 & \text{if } I_n = 1 \\ -1 & \text{if } I_n = 0. \end{cases}$$

- ~~Show~~ Draw a sample function / sequence of $D_n$




- Find $E[D_n]$, $var[D_n]$.