Filtering random processes in the frequeny domain (11-8)

$$X(t) \longrightarrow \boxed{h(t) \leftrightarrow H(f)} \longrightarrow Y(t)$$

WSS RP. LTI system. WSS RP

PSD $S_X(f) = FT[R_X(\tau)]$ PSD $S_Y(f) = FT[R_Y(\tau)]$.

Then : 2 ways of finding $R_Y(\tau)$:

1) $R_Y(\tau) = \int_{-\infty}^{+\infty} h(u) \int_{-\infty}^{+\infty} h(v) \, R_X(t+u-v) \, dv \, du$. (last class)

2) $S_Y(f) = S_X(f) \, |H(f)|^2$ where $|H(f)|^2 = H(f) H^*(f)$

complex conjugate.

Then $R_Y(\tau) = \text{inverse } FT[S_Y(\omega)]$

You will be given Table 11.1 in text if needed.

Application area: Entropy + Huffman coding.

Let's say we have 4 outcomes A, B, C, D and we want to produce a binary code for these (binary $\Leftrightarrow$ bits 0, 1).

Outcome		Codeword
A	$\Leftrightarrow$	$C(A) = 00$
B	$\Leftrightarrow$	$C(B) = 01$
C	$\Leftrightarrow$	$C(C) = 10$
D	$\Leftrightarrow$	$C(D) = 11$

This is one possible code for the symbols $\{A, B, C, D\}$

Is this code optimal (in terms of minimizing the expected ~~Cod~~ codeword length).

If $p(A) = p(B) = p(C) = p(D)$ then the above code is optimal.

But, if $p(A) = 1/2$, $p(B) = p(C) = 1/4$, $p(D) = 0$ then.

Symbol	Prob	Codeword
A	$1/2$	0
B	$1/4$	11
C	$1/4$	10
D	0	—

this is also optimal for these set of probabilities!

2 Questions:

Q1: ~~How~~ To how many bits/source symbol can I compress a source? X $H(X)$.

Q2: How do I find ~~these~~ optimal compression codewords? Huffman coding

minimizing the expected # bits/source symbol.

Entropy $H(X)$ of a RV X with ~~pdf~~ pmf $P_X(x)$

- we will only consider discrete sources
- entropy measures "how random" X is. The larger the entropy, the more random. Entropy is a number (positive number).
- assume we have an iid random sequence $X_0, X_1, X_2, \ldots$, where each X_i is distributed according to $p_X(x)$ (like RV X) $\forall i$.
- We call this iid sequence a "source"
- the entropy of a RV X with pmf (discrete) $P_X(x)$ is defined as:

$$H(X) \triangleq - \sum_{\underbrace{x \in S_X}_{\text{range of } X}} p_X(x) \log(p_X(x)) = - E_X[\log p_X(x)]$$

if log is base 2 then we talk about "bits"

Properties of entropy:

1) Entropy $H(X) \geq 0$, and $H(X) = 0$ if X is deterministic.
   (we assume $0 \log 0 = 0$).

   Proof: $0 \leq p_X(x) \leq 1$ so $\log p_X(x) \leq 0 \Rightarrow -\sum p(x) \log p_X(x) \geq 0$.

2) $H(X)$ does not depend on the actual values of the RV, on S_X, it just depends on $p_X(x)$.

   $S_X = \{A, B, C, D\}$ with probabilities $\{1/2, 1/4, 1/4, 0\}$
   $S_Y = \{\text{blue, red, green, yellow}\}$ with prob. $\{1/2, 1/4, 1/4, 0\}$
   $\Big\}$ same entropy!

3) $H(X) \leq \log \underbrace{|S_X|}_{\text{cardinality of } S_X}$ where S_X is range of X.

$|\{A, B, C, D\}| = 4$.

with equality if and ~~iff~~ only if X is uniform on S_X.

We will prove one direction, the other take ECE 534.

Prof: Let X be uniform on S_X. Then $p_X(x) = \dfrac{1}{|S_X|}$ for all $x \in S_X$.

Then $$H(X) = -\sum_{x \in S_X} p_X(x) \log p_X(x) = -\sum_{x \in S_X} \frac{1}{|S_X|} \log\left(\frac{1}{|S_X|}\right)$$

$$= \sum_{x \in S_X} \frac{1}{|S_X|} \log(|S_X|) = \log |S_X|$$