In the previous example, we used 2 concepts:

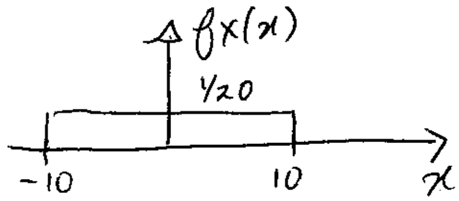
1. Functions of continuous RVs (quantizer)
2. Conditioning a continuous RV

Functions of a continuous RV

Let X be a continuous RV with pdf $f_X(x)$. Define $Y = g(X)$.
This derived RV / function of X may be discrete/continuous/mixed.
To obtain the pdf of a derived RV:

- Find CDF $F_Y(y) = P[Y \leq y]$ (usually in terms of $F_X(x)$)
- Compute PDF by calculating the derivative $f_Y(y) = \dfrac{dF_Y(y)}{dy}$

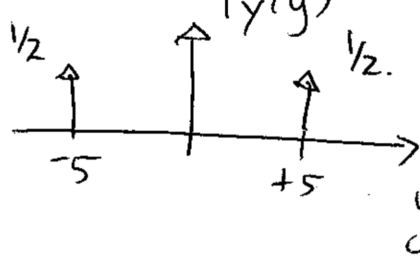
EX: X is uniform on $[-10, 10]$.



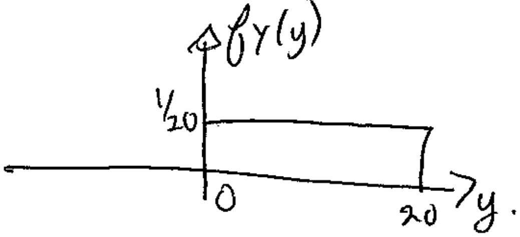
- Define Y the quantizer $Y = \begin{cases} -5 & X < 0 \\ +5 & X \geq 0 \end{cases}$

- $S_Y = \{-5, +5\}$, so Y is discrete.
- $P_Y(-5) = P[-10 \leq X < 0] = \int_{-10}^{0} \frac{1}{20} dx = \frac{10-0}{20} = 0.5$

  $P_Y(+5) = P[0 \leq X \leq 10] = \frac{1}{2}$.



- Define $Y = X + 10$, $S_Y = [0, 20] \Rightarrow$ continuous RV.



- $F_Y(y) = P[Y \leq y] = P[X + 10 \leq y] = P[X \leq y - 10]$
  $= F_X(y - 10)$

$\Rightarrow f_Y(y) = f_X(y - 10)$.

Conditioning a continuous RV

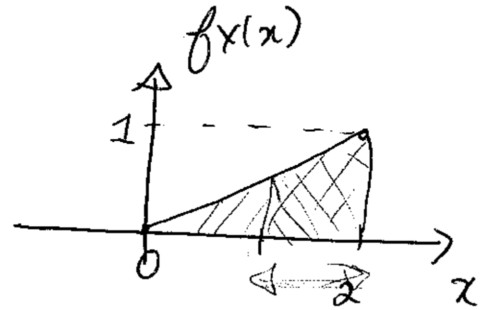
Continuous RV, X with pdf $f_X(x)$ and range S_X.
Let $B \subset S_X$ be an event with $P[B] > 0$.

- Conditional pdf given event B:

$$f_{X|B}(x) \triangleq \begin{cases} \dfrac{f_X(x)}{P[B]}, & x \in B \\ 0 & \text{else.} \end{cases}$$

- $$E[g(X)\,|B] \triangleq \int_{-\infty}^{+\infty} g(x)\, f_{X|B}(x)\, dx.$$

EX: $f(x) = \begin{cases} x/2 & 0 \le x \le 2. \\ 0 & \text{else.} \end{cases}$



Let $B = \{X \geq 1\} \Rightarrow P[B] = P[X \geq 1]$

$$= \int_1^2 f_X(u)\, du = \int_1^2 \frac{u}{2}\, du = \frac{3}{4}.$$

$$f_{X|B}(x) = \begin{cases} \dfrac{x/2}{3/4} = \dfrac{2x}{3} & 1 \le x \le 2. \quad (x \in B) \\ 0 & \text{else.} \end{cases}$$

$$E[X|B] = \int_{-\infty}^{+\infty} x\, f_{X|B}(x)\, dx = \int_1^2 x \cdot \frac{2x}{3}\, dx = \frac{2}{9} x^3 \Big|_1^2 = \frac{14}{9}.$$

<u>Thm</u>: If $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ then $Y = aX + b$ is again * Gaussian with mean $a\mu_X + b$, variance $a^2\sigma_X^2$.

$$Y \sim \mathcal{N}(a\mu_X + b,\ a^2\sigma_X^2).$$

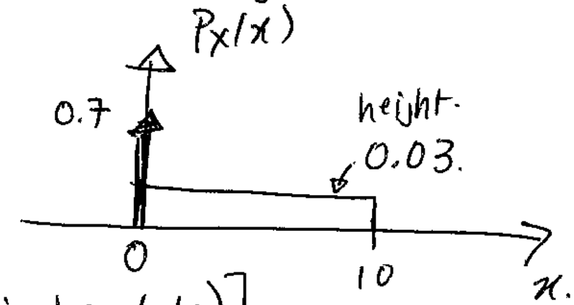
<u>Thm</u>: Given an event space $\{B_1, \ldots, B_m\}$ and the conditional PDFs $f_{X|B_i}(x)$, then

$$f_X(x) = \sum_{i=1}^{m} f_{X|B_i}(x)\, P[B_i]$$

Word problem: Textbook 3.6.9. (difficult). ◇ ⑦

For 70% of lectures, Prof. Y arrives on time. When Prof. Y is late, the arrival time delay is a continuous RV uniformly distributed from 0–10 minutes. Yet, as soon as Prof Y is 5 minutes late, all students get up + leave.
If a lecture starts when Prof. Y arrives + always ends 80 minutes after the scheduled starting time, what is the pdf of T, the length of time that the students observe a lecture.

X = Prof late arrival time.



Solution: ① What is S_T?

$T = \{0 \cup [75, 80]\} \Rightarrow$ mixed.

Easy case 1:

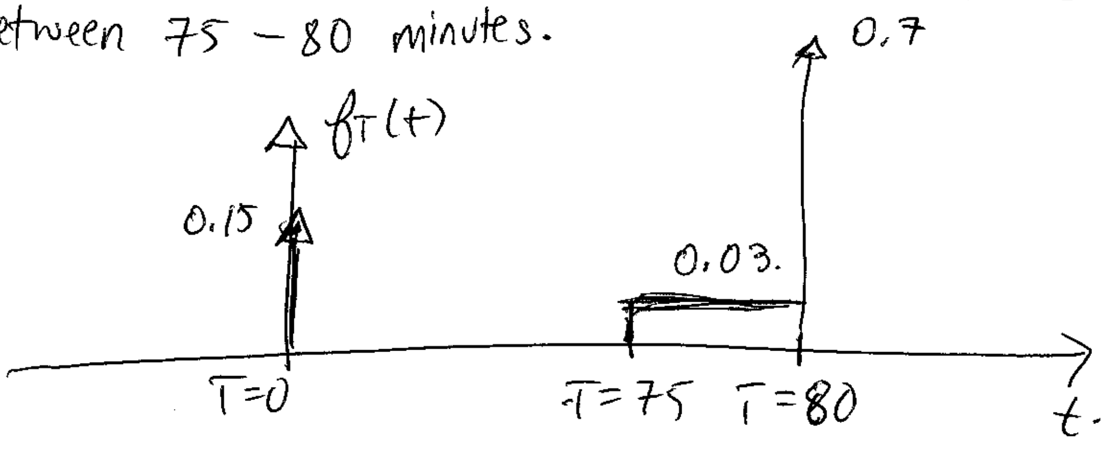
$P[T=0] = P[\text{Prof Y. is late} \cap (>5 \text{ minutes late})]$

$= P[X > 5]$.

$= \int_5^{10} (0.03)\,dx = 5 \times (0.03) = 0.15$.

$P[T = 80] = 0.7 = P[X = 0]$.

Intermediate cases $75 \le T < 80$: Lecture times are uniformly distributed between 75 – 80 minutes.



$$f_T(t) = \begin{cases} 0.15\,\delta(t) & t = 0. \\ 0.03 & 75 < t < 80 \\ 0.7\,\delta(t-80) & t = 80. \\ 0 & \text{else} \end{cases}$$