ASIDE: Theorem of Total Probability

For events: $P[A] = \sum_i P[A \mid B_i]\, P[B_i]$

for $\{B_1, B_2, \ldots, B_m\}$ an event space.

For CDFs: $F_X(u) = P[X \leq u]$.

$$F_X(u) = P[X \leq u] = \sum_i P[X \leq u \mid B_i]\, P[B_i]$$
$$= \sum_i F_{X|B_i}(u)\, P[B_i]$$

For PDFs: $f_X(u)$ (continuous) or $P_X(u)$ (discrete).

$$f_X(u) = \frac{d}{du} F_X(u) = \frac{d}{du}\left[\sum_i F_{X|B_i}(u)\, P[B_i]\right]$$
$$= \sum_i \frac{d}{du}\left[F_{X|B_i}(u)\right] P[B_i]$$
$$= \sum_i f_{X|B_i}(u)\, P[B_i]$$

Two Random Variables

$X \in \{1,2,3,4\}$, $Y \in \{1,2\}$

standing   # courses taken in summer.

$$P_X(x) = \begin{cases} 0.15 & x=1 \\ 0.25 & x=2 \\ 0.3 & x=3 \\ 0.3 & x=4 \\ 0 & \text{else} \end{cases} \qquad P_Y(y) = \begin{cases} 0.75 & y=1 \\ 0.25 & y=2 \\ 0 & \text{else} \end{cases}$$

One possible joint pdf $P_{XY}(x,y) \triangleq P[X=x, Y=y]$ consistent with the above $P_X(x)$, $P_Y(y)$ is:

Y \ X	1	2	3	4
2	0	0.05	0.1	0.1
1	0.15	0.2	0.2	0.2

$\equiv P_{XY}(x,y)$

$\boxed{\text{Sum of all elements in the table} = \underline{1}}$

"proper pdf"

- $P_{XY}(3,1) = P[X=3, Y=1] = 0.2$
- $P_X(3) = 0.3 = P_{XY}((X=3,Y=1) \cup (X=3,Y=2))$

$$= P_{XY}((X=3,Y=1)) + P_{XY}((X=3,Y=2)) \left(- \overbrace{\overbrace{P_{XY}((X=3,Y=1) \cap (X=3,Y=2))}^{\phi}}^{0} \right)$$

$$= 0.2 + 0.1$$

- $P_Y(1) = 0.2+0.2+0.2+0.15 = 0.75$

$$= P_{XY}[X = 1 \text{ or } 2 \text{ or } 3 \text{ or } 4,\ Y=1] = P_{XY}[(1,1) \cup (2,1) \cup (3,1) \cup (4,1)]$$

- Let B = event "student is (junior or senior) and (taking 1 course)".

$$P[B] = P[(3,1) \cup (4,1)] = P[(3,1)] + P[(4,1)] = 0.2 + 0.2 = 0.4$$

C = "student is (junior) or (senior and taking 1 course)"

$$P[C] = P[(3,1) \cup (3,2) \cup (4,1)] = 0.2 + 0.1 + 0.2 = 0.5$$

Theorem (Prob. of events)

For any discrete RVs X and Y with joint pmf $P_{XY}(x,y)$ and any set B in the X,Y plane (or S_{XY}), the probability of the event $\{(X,Y) \in B\}$ is

$$P[B] = \sum_{(x,y) \in B} P_{XY}(x,y)$$

Theorem (marginal pdfs)

We can find the marginal pdfs $P_X(x)$ and $P_Y(y)$ by summing over all points as follows:

$$P_X(x) = \sum_{y \in S_Y} P_{XY}(x,y) \qquad P_Y(y) = \sum_{x \in S_X} P_{XY}(x,y)$$

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constant.

Let $P_{XY}(x,y) = \begin{cases} c|x+y| & x \in \{-2,0,2\},\ y \in \{-1,0,1\} \\ 0 & \text{else} \end{cases}$

① Find c : Attack: make $P_{XY}(x,y)$ a "proper pdf" that sums to 1.

$$c\Big[|-2+(-1)| + |-2+0| + |-2+1| + |0+(-1)| + |0+0| + |0+1| + |2+(-1)| + |2+0| + |2+1|\Big] \overset{\text{WANT}}{=} 1$$

$$\Rightarrow c[14] = 1 \Rightarrow c = \frac{1}{14}.$$

Y \ X	−2	0	2
1	1/14	1/14	3/14
0	2/14	0	2/14
−1	3/14	1/14	1/14

• Find $P[Y \leq X] = P_{XY}\left[(0,-1) \cup (0,0) \cup (2,-1) \cup (2,0) \cup (2,1)\right]$

$$= P_{XY}[(0,-1)] + P_{XY}[(0,0)] + P_{XY}[(2,-1)] + P_{XY}[(2,0)] + P_{XY}[(2,1)]$$

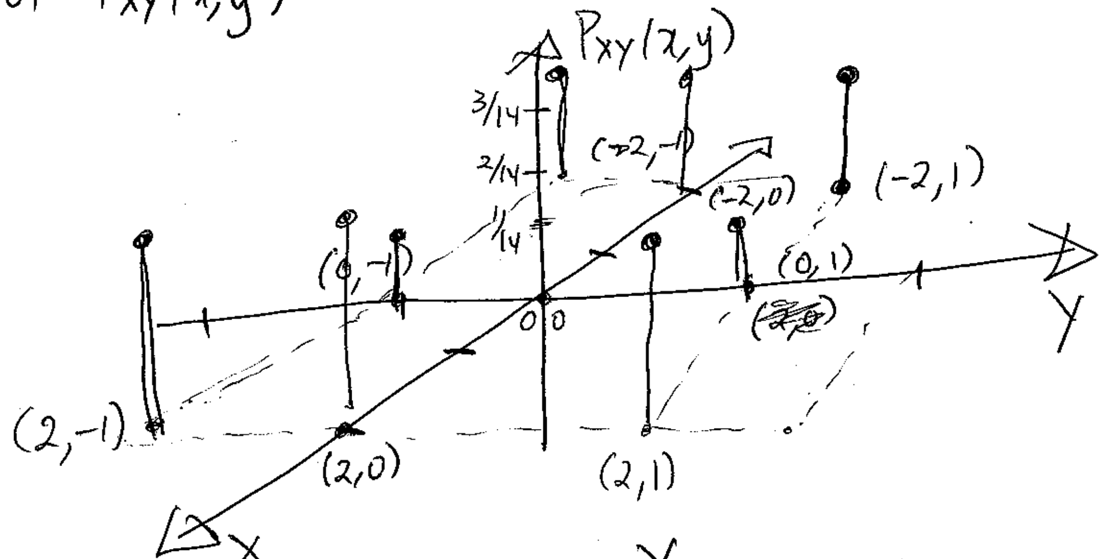
$$= \frac{1}{14} + \frac{0}{14} + \frac{1}{14} + \frac{2}{14} + \frac{3}{14} = \frac{1}{2}.$$

Find $P[Y > X] = 1 - P[Y \leq X] = \frac{1}{2}.$

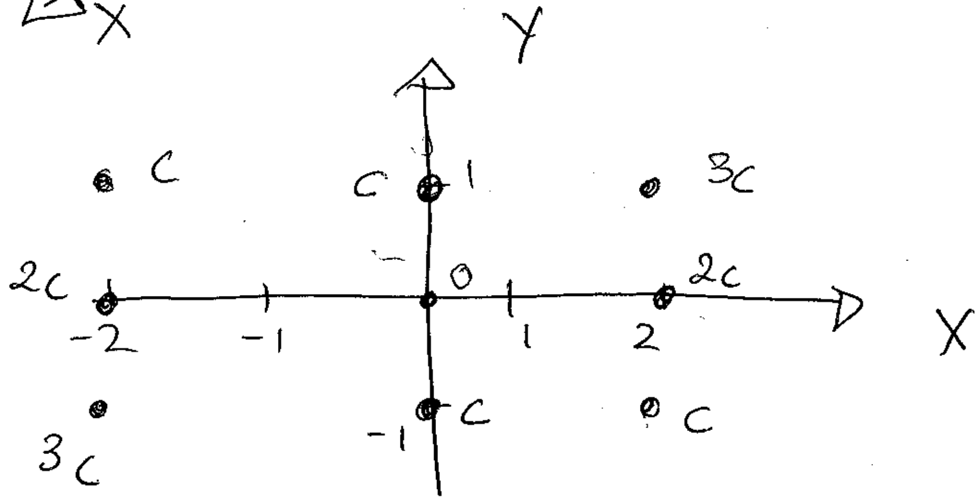
• Find $P[X = Y] = 0 = P_{XY}((0,0))$

• Find $P[X < 1] =$ sum of 1st + 2nd column $= 8/14$

• Plot $P_{XY}(x, y)$



Joint PDF

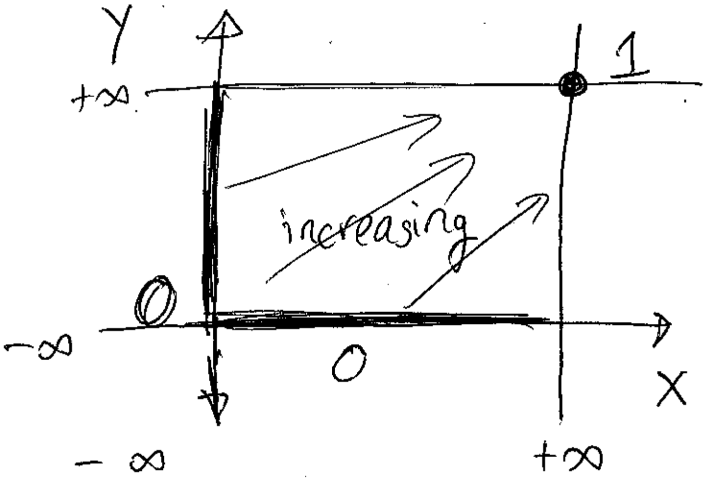


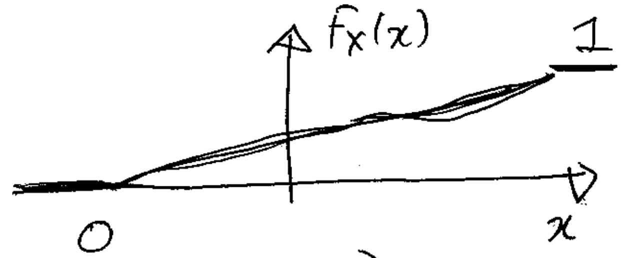
$(c = 1/14)$

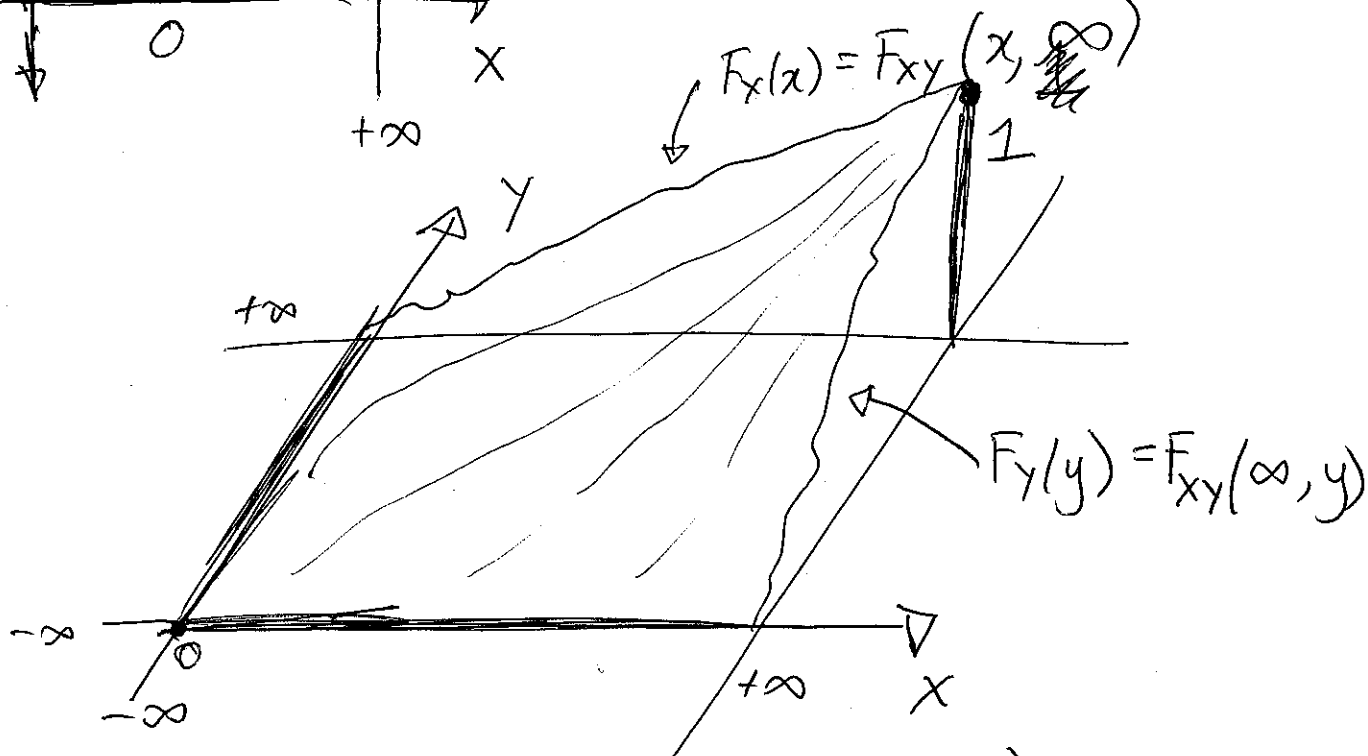
<u>Joint CDF</u>: of two random variables X, Y with PMF $P_{XY}(x,y)$ is

$$F_{XY}(x,y) = P[X \le x, Y \le y]$$

$$\left(F_X(x) = P[X \le x] \right)$$







<u>Properties of the CDF</u>: (deduce from definition of CDF)

- $0 \le F_{XY}(x,y) \le 1$
- $F_X(x) = F_{XY}(x, \infty) = P_{XY}[X \le x, Y \le \infty] = P_X[X \le x]$.
- $F_Y(y) = F_{XY}(\infty, y)$
- $F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = 0$
- $F_{XY}(\infty, \infty) = 1$.
- If $x_0 \le x_1$ and $y_0 \le y_1$ then $F_{XY}(x_0, y_0) \le F_{XY}(x_1, y_1)$