Two Continuous Random Variables

- Replace PMF with PDF, integrate insteam of sum!
- For the RVs X and Y, a probability model is given by the PDF function $\boxed{f_{XY}(x,y) \equiv \text{"joint PDF"}}$
- This joint PDF of continuous RVs X, Y is continuous and satisfies
  - $P[X \le x, Y \le y] = \underbrace{F_{XY}(x,y)}_{\text{"joint CDF"}} = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(u,v)\,du\,dv$
  - $f_{XY}(x,y) \geq 0 \quad \forall x, y$
  - $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx\,dy = 1$

  "proper pdf"
- For A an event in terms of continuous RVs X, Y (e.g. $A = \{\sqrt{X^2 + Y^3} \le 1\}$ then

$$P[A] = \iint_{A} f_{XY}(x,y)\,dx\,dy$$

integrate over the appropriate set!

- marginal PDFs

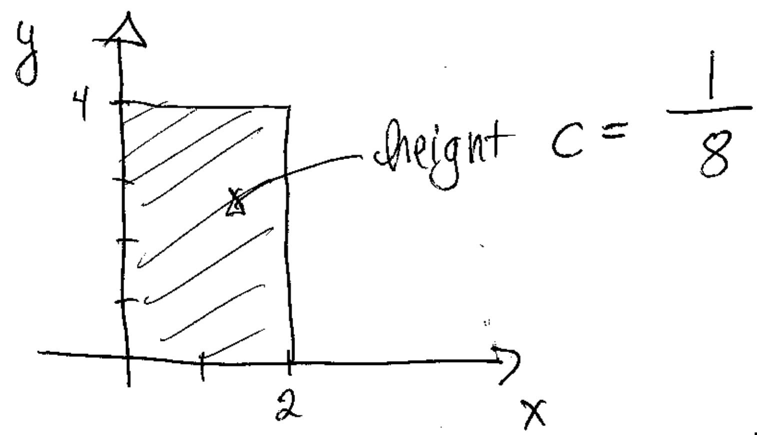
$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy \quad , \quad f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx$$

- $$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \, \partial y}$$

- $$E[g(X,Y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)\, f_{XY}(x,y)\, dx\, dy.$$

EX: X, Y are jointly uniform over a rectangle.

$$f_{XY}(x,y) = \begin{cases} c & 0<x<2,\ 0<y<4 \\ 0 & \text{else.} \end{cases}$$



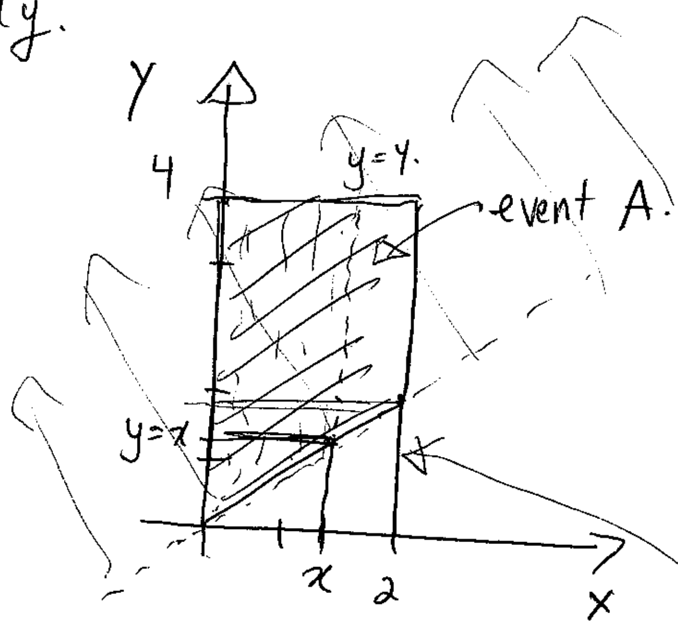
Find c: WANT $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY}(x,y)\, dx\, dy = 1$ WANT, solve for c.

$$\int_0^{4}\int_0^{2} \overbrace{f_{XY}(x,y)}^{c}\, dx\, dy = c(2)(4) \overset{\text{WANT}}{=} 1 \Rightarrow c = 1/8$$

• Find $P[A]$ when $A=\{Y>X\}$

→DRAW the area in the XY plane you're going to integrate over!

$$P[A] = \iint_A f_{XY}(x,y)\,dx\,dy.$$

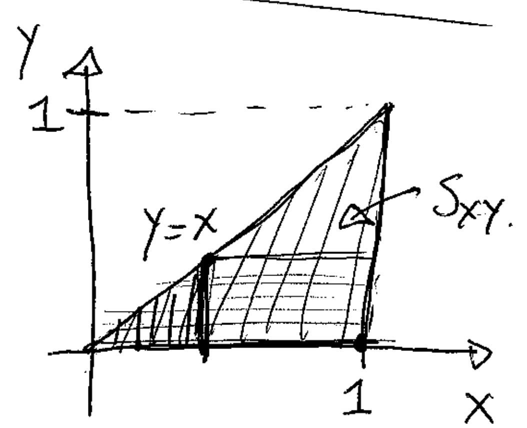


$$= \int_{x=0}^{2} \int_{y=x}^{4} \frac{1}{8}\,dy\,dx$$

$$= \int_{x=0}^{2} \left[\frac{1}{8}y\right]_x^4 dx$$

$$= \int_{x=0}^{2} \frac{1}{8}(4-x)\,dx = \frac{1}{2}x - \frac{1}{16}x^2 \Big|_0^2 = \frac{3}{4} = 1 - \underbrace{\left(\frac{2\times 2\times \frac{1}{8}}{2}\right)}_{\cancel{\text{area}}\ P[\text{triangle}]}$$

EX: $f_{XY}(x,y) = \begin{cases} Kx & 0 \le y \le x \le 1 \\ 0 & \text{else.} \end{cases}$



• Find K: (First, draw S_{XY})

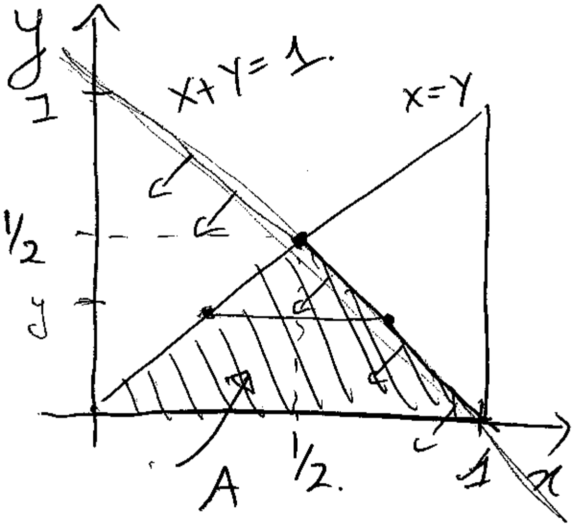
$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx\,dy \overset{\text{WANT, solve for K.}}{=} 1$$

$$\underbrace{\int_{x=0}^{1} \underbrace{\int_{y=0}^{x} (Kx)\,dy}\,dx} = \int_0^1 Kx\,[y]_0^x\,dx = \int_0^1 Kx^2\,dx = \frac{Kx^3}{3}\Big|_0^1 = \frac{K}{3}$$

$$\Rightarrow \boxed{K=3}$$

• Let $A = [X+Y \le 1]$. Find $P[A]$.

First, draw event A on XY plane.

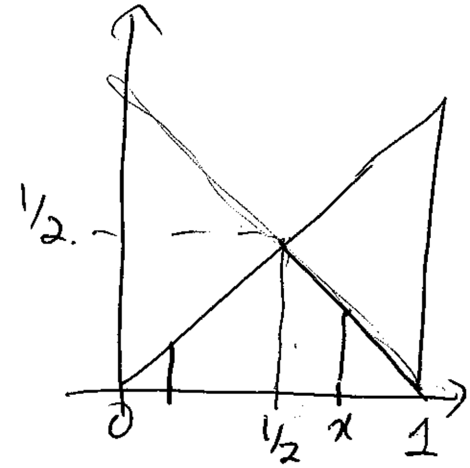


$$P[A] = \int_{y=0}^{1/2} \int_{x=y}^{x=1-y} \underbrace{(3x)}_{f_{XY}}\, dx\, dy$$

$$= \int_{y=0}^{1/2} \left[\frac{3x^2}{2}\right]_y^{1-y} dy = \frac{3}{2}\int_0^{1/2} (1-2y)\, dy = \cancel{0.25} \quad \frac{3}{8}$$

$$P[A] = \int_{x=0}^{1/2} \int_{y=0}^{y=x} 3x\, dy\, dx$$

$$+ \int_{x=1/2}^{1} \int_{y=0}^{y=1-x} 3x\, dy\, dx$$



$$= \int_0^{1/2} 3x^2 dx + \int_{1/2}^{1} 3x(1-x)\, dx = x^3\Big|_0^{1/2} + \frac{3x^2}{2}\Big|_{1/2}^{1} - x^3\Big|_{1/2}^{1}$$

$$= \left(\frac{1}{8}\right) + \frac{3}{2} - \frac{3}{2}\left(\frac{1}{4}\right) - 1 + \frac{1}{8}$$

$$= \frac{1}{2} + \frac{1}{8} - \frac{3}{8} + \frac{1}{8} = \frac{3}{8}$$