Functions of 2 RVs

Two RVs X and Y, are observed with either

→ joint PDF (continuous) $f_{XY}(x,y)$ Fully describe X and Y

→ joint PMF (discrete) $P_{XY}(x,y)$

Define $W = g(X,Y)$. What is the probability model for W?

For discrete: $W = g(X,Y)$ has pmf

$$P_W(w) = \sum_{x,y:\, g(x,y)=w} P_{XY}(x,y) \quad \text{for } w \in S_W$$

→ find this as a function of S_{XY}

For continuous: $W = g(X,Y)$ has CDF

$$F_W(w) = P[W \le w] = \iint_{g(x,y)\le w} f_{XY}(x,y)\,dx\,dy.$$

hard part.

$$f_W(w) = \frac{d}{dw}\left[F_W(w)\right].$$

EX: Machine transmits files of three sizes in bits: 700 kb, 1400 kb, 2800 kb.

⇒ RV $B \in S_B = \{700, 1400, 2800\}$ (kb)

The bit rates for transmission are three rates in kb/sec: 14 kb/sec, 28 kb/sec, 56 kb/sec

⇒ RV $R \in S_R = \{14, 28, 56\}$ (kb/sec)

The joint pdf of (B,R) is given by

B \ R	14 kb/s	28 kb/s	56 kb/s
700 kb	0.2 (50)	0.1 (25)	0
1400 kb	0.1 (100)	0.2 (50)	0.1 (25)
2800 kb	0	0.1 (100)	0.2 (50)

(T)

Find the expected transmission time/duration (in seconds).

<u>Sol</u>: Define new RV (transmission time) $T = B/R$.

<u>Want</u>: $E[T]$.

<u>Approach 1</u>: Find PMF of T, and then find $E[T] = \sum_{t \in S_T} t \, P_T(t)$

<u>Approach 2</u>: $E[B/R] = \sum_{(b,r) \in S_{BR}} b/r \, P_{BR}(b,r)$.

$$= \frac{700}{14}(0.2) + \frac{700}{28}(0.1) + \frac{1400}{14}(0.1) + \frac{1400}{28}(0.2) + \frac{1400}{56}(0.1)$$

$$+ \frac{2800}{28}(0.1) + \frac{2800}{56}(0.2) = 55 \text{ sec.}$$

<u>Back to approach 1, finding $P_T(t)$</u>:

$S_T = \{25, 50, 100\}$. $P_T(25) = \sum_{(b,r):\, b/r = 25} P_{BR}(b,r) = P[(700,28) \cup (1400,56)] = 0.1 + 0.1 = 0.2$

$P_T(50) = 0.2 + 0.2 + 0.2 = 0.6$

$P_T(100) = 0.1 + 0.1 = 0.2$.

$$E[T] = \sum_{t \in S_T} t \, P_T(t) = 25(0.2) + 50(0.6) + 100(0.2) = 55 \text{ (sec)}$$

<u>Expected values</u>:

(approach 2)

- Let $W = g(X,Y)$, then $E[W] = E[g(X,Y)] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x,y) \, P_{XY}(x,y)$.

- Note $E[X] = \sum_{x \in S_X} \sum_{y \in S_Y} x \, P_{XY}(x,y) = \sum_{x \in S_X} x \boxed{\sum_{y \in S_Y} P_{XY}(x,y)} = \sum_{x \in S_X} x \, P_X(x)$.

- If $g(X,Y) = g_1(X,Y) + g_2(X,Y) + \cdots + g_n(X,Y)$ & expectation is linear!

$$E[g(X,Y)] = E[g_1(X,Y)] + E[g_2(X,Y)] + \cdots + E[g_n(X,Y)]$$

E.g. $E[X+Y] = E[X] + E[Y]$. by linearity of expectation.

- $Var(X+Y) = ??$ Let $W = X+Y$, $\mu_W = \mu_X + \mu_Y$

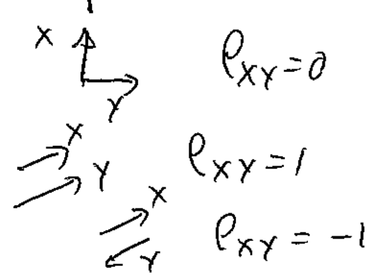
$$Var(X+Y) = Var(W) = E[(W-\mu_W)^2]$$
$$= E[(X+Y-\mu_X-\mu_Y)^2]$$
$$= \underbrace{E[(X-\mu_X)^2]}_{Var(X)} + \underbrace{E[(Y-\mu_Y)^2]}_{Var(Y)} + \underbrace{2E[(X-\mu_X)(Y-\mu_Y)]}_{2Cov(X,Y)}$$

- $Cov(X,Y) \triangleq E[(X-\mu_X)(Y-\mu_Y)]$ "covariance between X and Y"

$$= E[XY] - \mu_X E[Y] - \mu_Y E[X] + E[\mu_X \mu_Y]$$
$$= E[XY] - \mu_X \mu_Y - \mu_Y \mu_X + \mu_X \mu_Y$$
$$= \underbrace{E[XY]}_{\text{Correlation between } X, Y \ r_{XY}} - \mu_X \mu_Y$$

- $r_{XY} = E[XY] = cov(X,Y) + \mu_X \mu_Y$. "correlation btw. X and Y"
- If $cov(X,Y) = 0$: "X and Y are uncorrelated"
- If $r_{XY} = 0$: "X and Y are orthogonal"
- Correlation coefficient of two RVs X and Y is defined as

$$\rho_{XY} \triangleq \frac{cov(X,Y)}{\sqrt{var(X)\, var(Y)}}$$



<u>Claim</u>: $-1 \le \rho_{XY} \le +1$

<u>Proof</u>: $E\left[\left(\sigma_Y (X-\mu_X) \pm \sigma_X (Y-\mu_Y)\right)^2\right] \geq 0$

Why true? The term inside $E(\ )$ is ~~pos~~ non-negative, as it's $(\ )^2$.

$\sigma_Y = \text{std}(Y) = \sqrt{\text{var}(Y)}$, $\sigma_X = \text{std}(X) = \sqrt{\text{var}(X)}$.

$$\Leftrightarrow \quad \sigma_Y^2 E[(X-\mu_X)^2] + \sigma_X^2 E[(Y-\mu_Y)^2] \pm 2\sigma_X\sigma_Y E[(Y-\mu_Y)(X-\mu_X)] \geq 0.$$

using ($E[aX] = aE[X]$, $E[X+Y] = E[X] + E[Y]$)

$\text{Var}(aX) = a^2\,\text{Var}(X)$, $\text{Var}(aX+b) = a^2\,\text{Var}(X)$

definitions.

$$\Leftrightarrow \quad \sigma_Y^2\sigma_X^2 + \sigma_X^2\sigma_Y^2 \pm 2\sigma_X\sigma_Y \,\text{cov}(X,Y) \geq 0.$$

$$\Leftrightarrow \quad \frac{\pm\ \text{cov}(X,Y)}{\sigma_X\sigma_Y} \leq 1.$$

$$\Leftrightarrow \quad |\rho_{XY}| \le 1 \quad \Leftrightarrow \quad -1 \le \rho_{XY} \le +1.$$