Gaussian random variable

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

RV "is distributed as"

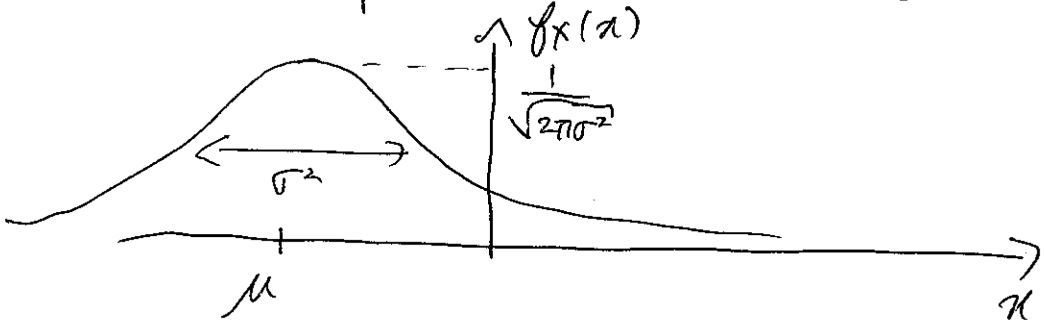
means X is distributed as a "Gaussian"/ "normal" random variable with mean μ and variance σ^2.

This means that the p.d.f. has the following form:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

A Gaussian is described by 2 parameters → mean μ
→ variance σ^2.

P.D.F. is a Bell-shaped curved around μ. Larger the σ^2, the wider the bell!



Note: $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1 = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx.$ → EXERCISE SHOW AT HOME!

Hint: change of variables from Cartesian to polar co-ordinates.

$$\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2/2}\,dx\right]^2 \overset{\text{SHOW}}{=} 1$$

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_{-\infty}^{+\infty} \frac{x-\mu+\mu}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx.$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} \underbrace{(x-\mu)e^{-\frac{(x-\mu)^2}{2\sigma^2}}}_{\text{is an odd function } f(y) = -f(-y).} dx + \mu \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx.$$

$$= 0 + \mu = \mu.$$

SHOW AT HOME.

$Var[x]$ can be shown to be σ^2 → show using integration by parts.

$X \sim \mathcal{N}(\mu, \sigma^2)$, then the "standard normal R.V."

$Z \sim \mathcal{N}(0, 1)$ and has $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

- the C.D.F. of a standard normal RV is denoted by.

$$\Phi(z) \triangleq F_Z(z) = \int_{-\infty}^{z} f_Z(t)\,dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\,dt.$$

- area under p.d.f. gives c.d.f.

$f_Z(z)$ $\Phi(z)$ is area under curve up until z.

$X \sim \mathcal{N}(\mu, \sigma^2)$

$$\Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\Rightarrow F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}}\,dt.$$

$$= \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{\sigma}{\sqrt{2\pi\sigma^2}} e^{-y^2/2}\,dy$$

change of variables
$$y = \frac{t-\mu}{\sigma}$$
$$dy = \frac{1}{\sigma}\,dt$$

$$= \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$\Phi(z)$ is only known numerically $\rightarrow$ use tables.

Marcum's Q-function

$Q(\cdot)$ = "standard normal complementary CDF"

$$Q(z) \triangleq P[Z \geq z] = \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} e^{-u^2/2}\,du.$$

"tail of the Gaussian"

Propeties of Q, Φ:

$$\Phi(-z) = 1 - \Phi(z), \quad Q(z) = 1 - \Phi(z).$$

EX: Let $X \sim N(\mu, \sigma^2)$.

- Find $P[\,|X-\mu| \le k\sigma\,]$ for $k=1,2,3 \dots$

$$P[|X-\mu| \le k\sigma] = P\left[-k \le \frac{X-\mu}{\sigma} \le k\right]$$

$$= P[-k \le Z \le k]$$ change variable $Z = \frac{X-\mu}{\sigma}$.

$$= \Phi(k) - \Phi(-k)$$ then $Z \sim N(0,1)$.

$$= 2\Phi(k) - 1.$$

For $k=1$: $P[-\sigma \le X-\mu \le \sigma] = 2\Phi(1) - 1 = 2(0.8413) - 1 = 0.6826$.

For $k=2$: $P[-2\sigma \le X-\mu \le 2\sigma] = 2\Phi(2) - 1 = 0.9545$

For $k=3$: $P[-3\sigma \le X-\mu \le 3\sigma] = 2\Phi(3) - 1 = 0.9973$.

EX: Exam grades in a course is modelled by a Gaussian R.V. $X \sim N(60, 400)$. The top 16% get A, next 22% get B. Where is the cutoffs z_1, z_2 between A/B and B/C?

$$P[X \ge z_1] = 0.16. \Rightarrow P\left[\frac{X-\mu}{\sigma} > \frac{z_1-\mu}{\sigma}\right] = 0.16.$$

$$= Q\left(\frac{z_1-\mu}{\sigma}\right) = 1 - \Phi\left(\frac{z_1-\mu}{\sigma}\right).$$

$\mu = 60$, $\sigma = 20$.

$$\Rightarrow x_1 = \mu + \sigma \underbrace{\Phi^{-1}(0.84)}_{0.9944} = 79.89 \approx 80.$$

SHOW: $z_2 = 66.1$.