P.M.F. $P[X=x] = P_X(x), \quad \forall x \in S_X$.

Cumulative Distribution Function (CDF)

For any real # x, the CDF is the probability that the RV X is no larger than x:

$$F_X(x) = P[X \le x] \quad , \text{ for } x \in S_X .$$

Properties of the CDF

For any discrete RV X with range $S_X = \{x_1, x_2, \dots\}$ satisfying

$x_1 \le x_2 \le x_3 \le \dots$

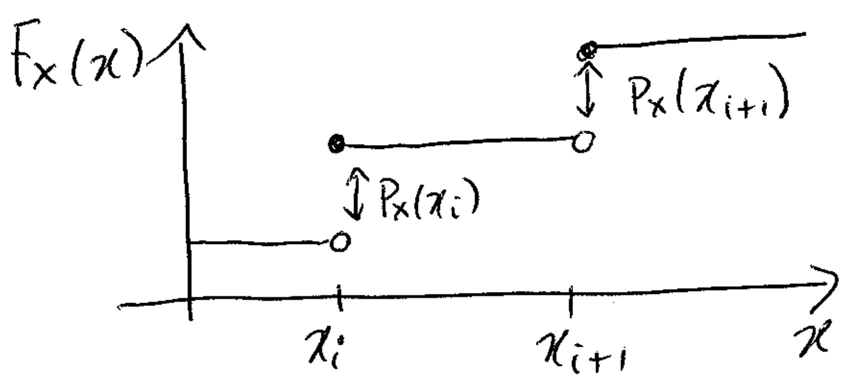
- $F_X(-\infty) = 0$

  $F_X(+\infty) = 1$

- $\forall x \le x'$, $\quad F_X(x) \le F_X(x')$

- ~~$\forall x_i \in S_X$ and $0 < \varepsilon < (x_i + x_{i+1})$~~

  ~~$F_X(x_i) - F_X(x_i + \varepsilon) = P_X(x_i)$~~



note!!

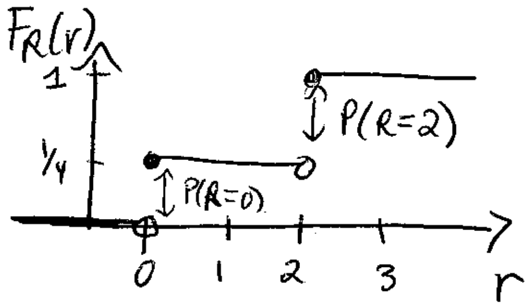
- $F_X(x) = F_X(x_i) \quad \forall x : \quad x_i \le x < x_{i+1}$

E.g. Consider the R.V. with p.m.f.

$$P_R(r) = \begin{cases} 1/4 & r=0 \\ 3/4 & r=2 \\ 0 & \text{else.} \end{cases}$$

Sketch the CDF of the R.V. R

$$F_R(r) = P[R \leq r] = \begin{cases} 0 & r<0 \\ 1/4 & 0 \leq r < 2 \\ 1 & r \geq 2. \end{cases}$$



Expected values/averages

Which bet do you take?

Make \$100 with prob. 1 in 10

Make \$1000 with prob. 1 in 1000 ?

What you did is compute the expectation!

The expected value of a R.V. X with p.m.f. $P_X(x)$ is.

"mean" $$\mu_X \triangleq E[X] \triangleq \sum_{x \in S_X} x \cdot P_X(x)$$ (weighted sum of outcomes)

E.g. roll a diee X = # dots face up. $P_X(x) = 1/6, \ x \in \{1,2,3,4,5,6\}$.

$$E[X] = Pr(X=1)\cdot 1 + Pr(X=2)\cdot 2 + \dots + Pr(X=6)\cdot 6$$
$$= \frac{1}{6}\left[1+2+3+4+5+6\right] = 3.5$$

Sample mean: $m_n = \frac{1}{n} \sum_{i=1}^{n} X_i$, $X_i \in S_X$. (X_i all distributed like X)

It may be shown that $\lim_{n\to\infty} m_n = E[X]$

E.g. What is the mean of a Bernouilli RV?

Bernouilli RV X has $S_X = \{0, 1\}$ with $P[X=1] = P_X(1) = p$.
for $0 \le p \le 1$. $P[X=0] = P_X(0) = 1-p$.

What is $E[X]$? $E[X] = \sum_{x \in S_X} x \cdot P_X(x)$

$$= 0 \cdot P_X(0) + 1 \cdot P_X(1)$$
$$= 0 + 1 \cdot p$$
$$= p.$$

E.g. What is the mean of a geometric RV? repeated, independent Bernouilli RVs.

Geometric R.V. is the position of the 1st success (1) in a Bernouilli trial.

$$\underset{P[X=x]}{P_X(x)} = \begin{cases} p(1-p)^{x-1} & x = 1, 2, 3, \ldots \\ 0 & \text{else.} \end{cases}$$

$$S_X = \{1, 2, 3, \ldots \}$$

WHY?

$$E[X] = \sum_{x \in S_X} x \cdot P_X(x) = \sum_{x=1}^{\infty} x p (1-p)^{x-1} = \frac{1}{p}$$

Finite sum of series $K, K\alpha, K\alpha^2, K\alpha^3, \dots$

$$S_m = K + K\alpha + K\alpha^2 + \cdots + K\alpha^{m-1}$$

$$\alpha S_m = \quad K\alpha + K\alpha^2 + \cdots + K\alpha^{m-1} + K\alpha^m$$

$$S_m - \alpha S_m = K - K\alpha^m$$

$$\Rightarrow \quad S_m = K \frac{(1-\alpha^m)}{(1-\alpha)} \quad \text{if} \quad \alpha \neq 1$$

$$\Rightarrow \quad \sum_{n=0}^{m-1} K\alpha^n = K \cdot \frac{1-\alpha^m}{1-\alpha} \quad \text{for} \quad \alpha \neq 1$$

$\Rightarrow$ If $|\alpha| < 1$ then $\displaystyle \lim_{m\to\infty} S_m \triangleq S_\infty = \sum_{n=0}^{\infty} K\alpha^n = \frac{K}{1-\alpha}.$

<u>Want</u>: $\displaystyle \sum_{x=1}^{\infty} x p (1-p)^{x-1}$

<u>Know</u>: $\displaystyle \sum_{k=0}^{\infty} (1-p)^k = \frac{1}{1-(1-p)} = \frac{1}{p}.$

So, $\displaystyle \frac{d}{dp}\left[\sum_{k=0}^{\infty} (1-p)^k\right] = \sum_{k=1}^{\infty} k(1-p)^{k-1}(-1) = \frac{d}{dp}\left[\frac{1}{p}\right] = -\frac{1}{p^2}$

$$\Rightarrow \quad \boxed{\sum_{x=1}^{\infty} x p (1-p)^{x-1} = \frac{1}{p}.}$$