E.g. A message is sent over a channel repeatedly with probability 0.05 of a successful reception. (1)

a) On average, how many transmissions are needed for the message to be successfully received?

Let X = random variable which takes on value of the # of transmiss until successful reception.

$S_X = \{1, 2, 3, \ldots\}$

So, X is Geometrically distributed! is distributed according to

Last class, we saw that if $X \sim \text{Geometric}(p)$

Then $E[X] = \frac{1}{p} = \frac{1}{(0.05)} = 20.$

b) Prob[message is received in 20 transmissions or less]?

$$P[X \leq 20] = 1 - P[X > 20]$$

$$= 1 - (0.95)^{20} = 0.642.$$

E.g. Find the mean of a binomial RV

The binomial R.V. counts the # of successes in n Bernoulli trials.

$S_X = \{0, 1, 2, \ldots n\}$. $P_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x \in S_X \\ 0 & \text{else.} \end{cases}$

$x \in S_X$

$$E[X] = \sum_{x \in S_X} x P_X(x) = \sum_{x=0}^{n} x \cdot \binom{n}{x} \cdot p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

why?

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x}$$

Let $y = x-1$, let $m = n-1$, and substitute $x = y+1$, $n = m+1$, noting that the limits $x=1 \Rightarrow y=0$, and $x=n \Rightarrow y=m$

$$E[X] = \sum_{y=0}^{m} \frac{(m+1)!}{y!(m-y)!} p^{y+1} (1-p)^{m-y}$$

$$= (m+1)p \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^y (1-p)^{m-y}$$

$$= np \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^y (1-p)^{m-y} = 1$$

$$= np.$$

recognize as p.m.f. of binomial R.V.

OR use Binomial Thm.

<u>Binomia Thm</u>: $(a+b)^m = \boxed{\sum_{y=0}^{m} \frac{m!}{y!(m-y)!} a^y b^{m-y}}$

Take $a=p$, $b=1-p$, The RHS same, $(p+(1-p))^m = 1^m = \underline{1}$.

So, $E[X] = np$.

E.g. Mean of a Poisson RV

The Poisson RV corresponds to the # of arrivals in time T with arrival rate λ.

$$S_X = \{0, 1, 2, \ldots\}.$$

$$P_X(x) = \frac{\alpha^x e^{-\alpha}}{x!} \qquad x = 0, 1, 2, \ldots \qquad \text{for } \alpha = \lambda T$$

We show $E[X] = \alpha = \lambda T$

$$E[X] = \sum_{x=0}^{\infty} x P_X(x) = \sum_{x=0}^{\infty} x \frac{\alpha^x}{x!} e^{-\alpha}$$

$$= \alpha \sum_{x=1}^{\infty} \frac{\alpha^{x-1}}{(x-1)!} e^{-\alpha}$$

define $l = x-1$

$$= \alpha \underbrace{\sum_{l=0}^{\infty} \frac{\alpha^l}{l!} e^{-\alpha}}_{1} = \alpha$$

as $\sum_{l=0}^{\infty} \frac{\alpha^l}{l!} = e^{\alpha}$.

Functions of a random variable / derived RV

Given: RV X with pmf. $P_X(x)$, and range $S_X(x)$

Define: $Y = g(X)$, a new RV.

Q: What is the pmf. of Y, $P_Y(y)$?

Approach:
- Find range S_Y of Y
- For each y in S_Y, find $P_Y(y) = P[Y = y]$.

$$P_Y(y) = \sum_{x : g(x) = y} P_X(x)$$