Gaussian in noise

$X \in \{-A, +A\}$ with prob $P_X(A) = 1/2$, $P_X(-A) = 1/2$

Then it passes through an AWGN channel and we receive
"additive ~~white~~ Gaussian noise"

$$Y = X + N \quad \text{where } N \sim \mathcal{N}(0, \sigma_N^2)$$

1. What is pdf of Y
2. What is the probability of error as a function of A, σ_N^2?
3. What is a decision rule for deciding whether $+A$ or $-A$ was sent.

① pdf * $f_Y(y) = f_{Y|X=-A}(y|X=-A)\,p(X=-A) + f_{Y|X=+A}(y|X=+A)\,p(X=+A)$

cdf. $F_Y(y) = F_{Y|X=-A}(y|X=-A)\,P(X=-A) + F_{Y|X=+A}(y|X=+A)\,p(X=+A)$

$$E[Y] = E[Y|X=+A]\,P[X=+A] + E[Y|X=-A]\,P[X=-A].$$

* I like this one as $p(X=-A) = p(X=+A) = 1/2$ (GIVEN)
and I can recognize

$$f_{Y|X=+A}(y|X=+A) \sim N(+A, \sigma_N^2) = \frac{1}{\sqrt{2\pi\sigma_N^2}} e^{-\frac{(y-A)^2}{2\sigma_N^2}}$$

$$Y = +A + N, \quad N \sim \mathcal{N}(0, \sigma_N^2) \Rightarrow Y \sim \mathcal{N}(+A, \sigma_N^2)$$

$$f_{Y|X=-A}(y|X=-A) \sim \mathcal{N}(-A, \sigma_N^2) = \frac{1}{\sqrt{2\pi\sigma_N^2}} e^{-\frac{(y+A)^2}{2\sigma_N^2}}$$

(1) What is the pdf of Y?

$$f_Y(y) = \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma_N^2}}e^{-\frac{(y-A)^2}{2\sigma_N^2}} + \frac{1}{2\sqrt{2\pi\sigma_N^2}}e^{-\frac{(y+A)^2}{2\sigma_N^2}}$$

(3) One good decision rule is decide +A sent if $Y>0$.
decide −A sent if $Y<0$.

(2) Can now calculate the probability of error for this decision rule!

$$P[\text{error}] = P[\text{decide } -A \text{ sent} \mid +A \text{ sent}]\,P[+A \text{ sent}] +$$
$$P[\text{decide } +A \text{ sent} \mid -A \text{ sent}]\,P[-A \text{ sent}].$$
$$= P[Y<0 \mid +A \text{ sent}]\,P[+A \text{ sent}] +$$

this is a conditional CDF

$$P[Y>0 \mid -A \text{ sent}]\,P[-A \text{ sent}].$$

this is a conditional ~~CDF~~ 1 − CDF

$$= \cancel{2P[Y<0 \mid +A \text{ sent}]}$$
$$= 2\,P[Y>0 \mid -A \text{ sent}]\,P[-A \text{ sent}].$$
$$= 2\,P[-A+N>0]\,\tfrac{1}{2} = P[N>+A].$$

Gaussian.

$$= P\left[\frac{N}{\sigma_N} > \frac{+A}{\sigma_N}\right].$$
$$= Q\left(\frac{A}{\sigma_N}\right) = Q\left(\sqrt{SNR}\right) \qquad SNR = \frac{E[X^2]}{E[N^2]} = \frac{A^2}{\sigma_N^2}.$$

For Gaussian, we donot have a closed form expression for the CDF!! (we do for the pdf).

$$\Phi(x) \triangleq \int_{-\infty}^{x} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{(u)^2}{2}}}_{\mathcal{N}(0,1)} du. \rightarrow \text{look up tables!!}$$

"standard normal"

$$Q(x) \triangleq \int_{x}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}}_{\mathcal{N}(0,1).} du. \rightarrow \text{look up tables!!}$$

Let's say I want to ensure $P[\text{error}] < 10^{-5}$, what value of A should I transmit at?

$$P(\text{error}) = Q\left(\frac{A}{\sigma_N}\right) \overset{\text{WANT}}{=} 10^{-5}$$

x	Q(x)
	⋮
3.41	10^{-5}

$\Rightarrow$ I need $\frac{A}{\sigma_N} = 3.41$ by looking at table

$\Rightarrow \quad A = 3.41 \times \sigma_N.$