Then,

$$f_{X,Y|B}(x,y) = \begin{cases} 2/15 & \text{if } 0 \le x \le 5,\ 0 \le y \le 3,\ x+y \ge 4 \\ 0 & \text{else.} \end{cases}$$

EX: Toss a coin and roll a die.

Sample space : $\{h1, h2, h3, h4, h5, h6, t1, t2, t3, t4, t5, t6\}$

Let X = # heads on cointoss. $S_X = \{0, 1\}$

Y = # dots facing up + # heads on cointoss $S_Y = \{1,2,3,4,5,6,7\}$.

• $P_{XY}(x,y)$

X \ Y	1	2	3	4	5	6	7
0	1/12 (t1)	1/12 (t2)	1/12	1/12	1/12	1/12	0
1	0 (ϕ)	1/12 (t2)	1/12	1/12	1/12	1/12	1/12

• $P_X(x) = \begin{cases} 1/2 & x = 0, 1 \\ 0 & \text{else} \end{cases}$, $P_Y(y) = \begin{cases} 1/6 & y = 2,3,4,5,6 \\ 1/12 & y = 1, 7 \\ 0 & \text{else.} \end{cases}$

• $$\boxed{P_{Y|X=x}(y) \triangleq \frac{P_{XY}(x,y)}{P_X(x)} = P_{Y|X}(y|x)} \quad \text{Conditional pmf.}$$

• If $x = 0 \Rightarrow P_{Y|X=0}(y) = \begin{cases} \dfrac{P_{XY}(0,y)}{P_X(0)} = 1/6 & y = 1, 2, \ldots, 6. \\ 0 & \text{else} \end{cases}$

If $x = 1 \Rightarrow P_{Y|X=1}(y) = \begin{cases} \dfrac{P_{XY}(1,y)}{P_X(1)} = 1/6 & y = 2, 3, 4, \ldots, 7 \\ 0 & \text{else.} \end{cases}$

$$\boxed{E[Y|X=x] = \sum_{y \in S_{Y|X=x}} y \, P_{Y|X}(y|x).}$$

- $E[Y|X=0] = \sum_{y=1}^{6} y \cdot \frac{1}{6} = \frac{7}{2}$, $E[Y|X=1] = \sum_{y=2}^{7} y \cdot \frac{1}{6} = \frac{9}{2}$.

$$\boxed{Var[Y|X=x] = E[Y^2|X=x] - (E[Y|X=x])^2}$$

- $Var(Y|X=0) = \sum_{y=1}^{6} y^2 \cdot \frac{1}{6} - \left(\frac{7}{2}\right)^2$.

Independent RVs

Events A, B independent if: $P[A \cap B] = P[A] P[B]$.

Discrete RVs X, Y independent if: $\forall x \in S_X, \forall y \in S_Y$

$$P[X=x, Y=y] = P[X=x] P[Y=y]$$

$$\Rightarrow * \boxed{P_{XY}(x,y) = P_X(x) P_Y(y).} \; * \text{ FOR INDEPENDENT}$$

Continuous RVs X, Y independent if $\boxed{f_{XY}(x,y) = f_X(x) f_Y(y)}$

$$\boxed{P_{XY}(x,y) = P_{Y|X}(y|x) P_X(x) \quad \text{HOLDS ALWAYS}}$$

If X, Y are independent then $P_{XY}(x,y) = P_X(x) P_Y(y)$ so,

$$\boxed{P_{Y|X}(y|x) = P_Y(y) \quad \text{FOR INDEPENDENT } X, Y}$$

Ex: X, Y with the following joint PDF are independent?

		$Y=1$	$Y=2$
X	1	1/4	1/4
	2	1/4	1/4

✓ independent

$P_X(x) = \{ 1/2 \quad x = 1,2$

$P_Y(y) = \{ 1/2 \quad y = 1,2$

$P_{XY}(x,y) = P_X(x)\, P_Y(y) \quad \forall x, y.$

		$Y=1$	$Y=2$
X	1	1/2	0
	2	0	1/2

X not independent

$P_X(x) = \{ 1/2 \quad x = 1,2$

$P_Y(y) = \{ 1/2 \quad y = 1,2$

But $P_{XY}(x,y) \neq P_X(x) P_Y(y)$

Consequences of independence: (only true for X, Y independent!)

- $E[XY] = r_{XY} = E[X]E[Y] = \mu_X \mu_Y$

  Proof: $E[XY] \overset{\text{always}}{=} \iint x \cdot y \, \boxed{f_{XY}(x,y)} \, dx\, dy.$ ↓ independence

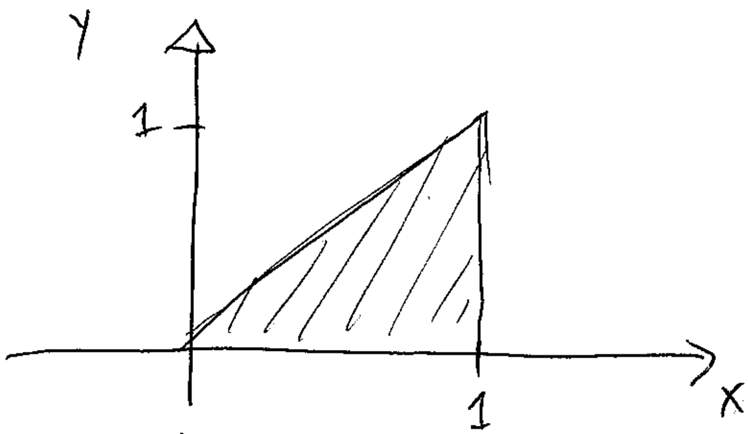
  $= \iint x \cdot y \, \boxed{f_X(x)\, f_Y(y)} \, dx\, dy$

  $= \int x f_X(x)\, dx \cdot \int y f_Y(y)\, dy.$

  $= \mu_X \cdot \mu_Y = E[X] \cdot E[Y]$

- $E[Y \mid X = x] = E[Y] \quad \forall x \in S_X.$
- $Var(X+Y) = Var(X) + Var(Y)$
- $Cov(X,Y) = 0 = E[XY] - \mu_X \mu_Y$
- $E[g(X) h(Y)] = E[g(X)]\, E[h(Y)].$

EX: $f_{XY}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1 \\ 0 & \text{else.} \end{cases}$



- Find $f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx = \begin{cases} 0 & y<0,\ y>1 \\ \int_{x=y}^{1} 2\,dx = 2(1-y) & \text{else.} \end{cases}$

- $f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy = \begin{cases} 0 & x<0,\ x>1 \\ \int_{y=0}^{x} 2\,dy = 2x & \text{else.} \end{cases}$

So, $f_Y(y)\, f_X(x) = \begin{cases} 4x(1-y) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{else.} \end{cases}$

$\neq f_{XY}(x,y)$ so <u>NOT</u> independent!